Math 662 **Spring 2014**
Homework 2 **Drew Armstrong**

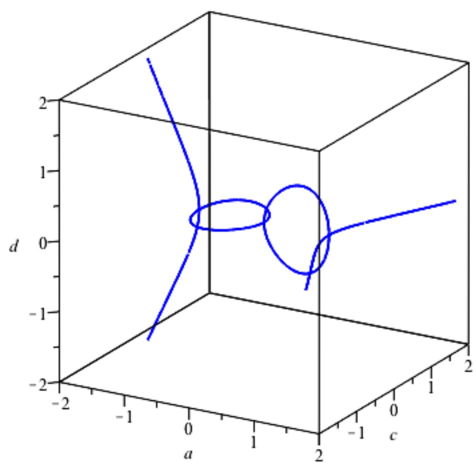
Problem 0. **(Drawing Pictures)** The equation $y^2 = x^3 - x$ defines a "curve" in the complex "plane" $\mathbb{C}^2$. What does it look like? Unfortunately we can only see real things, so we substitute $x = a + ib$ and $y = c + id$ with $a, b, c, d \in \mathbb{R}$. Equating real and imaginary parts then gives us **two simultaneous equations**:

$$a^3 - a - 3ab = c^2 - d^2, \tag{1}$$

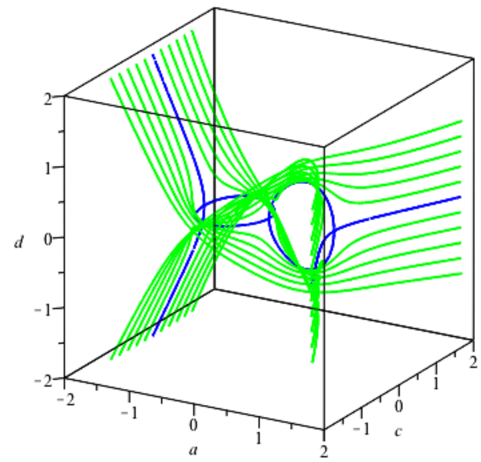
$$b^3 + b - 3a^2 b = -2cd. \tag{2}$$

These equations define a real 2-dimensional surface in real 4-dimensional space $\mathbb{R}^4 = \mathbb{C}^2$. Unfortunately we can only see 3-dimensional space so we will interpret the b coordiante as "time". Sketch the curve in real (a, c, d)-space at time $b = 0$. [Hint: It will look 1-dimensional to you.] Can you imagine what it looks like at other times b?

At time $b = 0$ equation (2) becomes $0 = cd$ which implies that $c = 0$ or $d = 0$. When $d = 0$ equation (1) becomes $a^3 - a = c^2$. This is a curve in the (a, c)-plane which we sketched on HW1. When $c = 0$ equation (1) becomes $a^3 - a = -d^2$, or $(-a)^3 - (-a) = d^2$. This curve lives in the (a, d)-plane. It looks just like the curve in the (a, c)-plane but it is reflected across the (c, d)-plane and rotated 90°. The full solution is the disjoint union of these two:



It looks like four circles glued together in a chain. If you can imagine, the two outer circles meet at the point at infinity, and the four circles together form the skeleton of a torus. The other times sweep out the surface of the torus. For example, here I have plotted the curves in (a, c, d)-space for $b \in \{-.5, -.4, -.3, -.2, -.1, 0, .1, .2, .3, .4, .5\}$:



Problem 1. (Local Rings) Let R be a ring. We say R is local if it contains a unique (nontrivial) maximal ideal.

(a) Prove that R is local if and only if its set of non-units is an ideal.

(b) Given a prime ideal $P \leq R$, prove that the localization

$$R_P := \left\{ \frac{a}{b} : a, b \in R, b \notin P \right\}$$

is a local ring. [Hint: The maximal ideal is called PR_P.]

(c) Consider a prime ideal $P \leq R$. By part (b) we can define the residue field R_P/PR_P. Prove that we have an isomorphism of fields:

$$\text{Frac}(R/P) \approx R_P/PR_P.$$

Proof. For part (a), let $M \subseteq R$ denote the set of non-units. Note that $M \neq R$ because $1 \notin M$. First we assume that M is an ideal. In this case, let $I \leq R$ be any ideal of R not contained in M. Then by definition I contains a unit and hence $I = R$. (If $u \in I$ is a unit then $u \in I$ and $u^{-1} \in R$ imply $1 = uu^{-1} \in I$. Then for all $r \in R$ we have $r = 1r \in I$.) We conclude that M is the unique maximal ideal of R, hence R is local. Conversely, assume that R is local with unique maximal ideal $\mathbf{m} < R$. Since $\mathbf{m} \neq R$ we know that $\mathbf{m}$ contains no units, hence $\mathbf{m} \subseteq M$. On the other hand, we will show that $M \subseteq \mathbf{m}$. Suppose for contradiction that there exists $x \in M$ with $x \notin \mathbf{m}$. Since $x \notin \mathbf{m}$ and $R/\mathbf{m}$ is a field ($\mathbf{m}$ is maximal) there exists $y \in R$ such that

$$xy + \mathbf{m} = (x + \mathbf{m})(y + \mathbf{m}) = 1 + \mathbf{m}.$$

This implies that $xy = 1+a$ for some $a \in \mathbf{m}$. But then $1+a \notin \mathbf{m}$ since otherwise $1 = (1+a)-a$ is in $\mathbf{m}$ (this is a contradiction because $\mathbf{m} \neq R$). Hence the ideal $(xy) = (1+a)$ strictly contains $\mathbf{m}$ and since $\mathbf{m}$ is maximal this implies $(xy) = R$. We conclude that xy is a unit, hence x is a unit: $x(y(xy)^{-1}) = (xy)(xy)^{-1} = 1$. This contradicts the fact that $x \in M$ and we conclude that $M = \mathbf{m}$ is an ideal.

[Remark: I could have given a shorter proof of $M \subseteq \mathbf{m}$ as follows. Consider any $x \in M$. Since $(x) < R$ is a proper ideal, it is contained in some proper maximal ideal, hence $(x) \leq m$. We conclude that $x \in \mathbf{m}$. But this argument implicitly uses the Axiom of Choice. The proof I gave above shows that the Axiom of Choice is not necessary.]

For part (b), let $P \leq R$ be prime and consider the localization

$$R_P := \left\{ \frac{a}{b} : a, b \in R, b \notin P \right\}.$$

I will show that the nonunits of R_P form an ideal. We can think of R_P as a subring of $\mathrm{Frac}(R)$. Let $\frac{a}{b} \in R_P$. Since $b \neq 0$ this fraction has inverse $\frac{b}{a} \in \mathrm{Frac}(R)$. This inverse will be in R_P if and only if $a \notin P$. In other words, $\frac{a}{b} \in R_P$ is a nonunit if and only if $a \in P$. Let

$$PR_P := \left\{ \frac{a}{b} : a, b \in R, a \in P, b \notin P \right\}$$

denote the set of nonunits. This is an ideal because given $\frac{a}{b}, \frac{c}{d} \in PR_P$ and $\frac{e}{f} \in R_P$ (i.e. with $a, c, e \in P$ and $b, d, f \notin P$) we have

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd} \in PR_P$$

because $ad - bc \in P$ and $bd \notin P$, and

$$\frac{a}{b} \cdot \frac{e}{f} = \frac{ae}{bf} \in PR_P$$

because $ae \in P$ and $bf \notin P$. We conclude that R_P is local with maximal ideal PR_P.

For part (c), note that R_P/PR_P is a field because $PR_P < R_P$ is a maximal ideal. Note also that R/P is a domain because $P < R$ is a prime ideal, thus we can form the field of fractions $\mathrm{Frac}(R/P)$. I claim that these two fields are isomorphic. To see this, note first that $s + P \neq 0 + P$ implies $s \notin P$. Thus we can define a map from $\mathrm{Frac}(R/P)$ to R_P/PR_P by

$$\frac{r+P}{s+P} \longmapsto \frac{r}{s} + PR_P. \tag{3}$$

To see that this is well-defined, consider $s, u \notin P$ and suppose that $\frac{r+P}{s+P} = \frac{t+P}{u+P}$, i.e., $ru + P = (r+P)(u+P) = (s+P)(t+P) = st + P$. Then since $ru - st \in P$ and $su \notin P$ we conclude that $\frac{r}{s} - \frac{t}{u} = \frac{ru - st}{su} \in PR_P$. It is easy to see that the map is a surjective ring homomorphism (details omitted). Finally we will show that the map is injective by showing that the kernel is trivial. Consider $\frac{r}{s} \in R_P$ (i.e. with $s \notin P$) and suppose that

$$\frac{r}{s} + PR_P = PR_P,$$

i.e., that $\frac{r}{s} \in PR_P$. This means that $r \in P$ and hence $\frac{r+P}{s+P}$ is the zero element of $\mathrm{Frac}(R/P)$. We conclude that

$$\mathrm{Frac}(R/P) \approx R_P/PR_P.$$

$\square$

Problem 2. (Formal Power Series) Let K be a field and consider the ring of formal power series:

$$K[[x]] := \left\{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots : a_i \in K \text{ for all } i \in \mathbb{N} \right\}.$$

The "degree" of a power series does not necessarily exist. However, for all nonzero $f(x) = \sum_k a^k x^k$ we can define the "order" $\mathrm{ord}(f) :=$ the minimum k such that $a_k \neq 0$.

(a) Prove that $K[[x]]$ is a domain.

(b) Prove that $K[[x]]$ is a Euclidean domain with norm function $\mathrm{ord} : K[[x]] - \{0\} \to \mathbb{N}$. (You can define $\mathrm{ord}(0) = -\infty$ if you want.) [Hint: Given $f, g \in K[[x]]$ we have $f|g$ if and only if $\mathrm{ord}(f) \leq \mathrm{ord}(g)$, so the remainder is always zero.]
(c) Prove that the units of $K[[x]]$ are just the power series with nonzero constant term.
(d) Conclude that $K[[x]]$ is a local ring.
(e) Prove that $\mathrm{Frac}\,(K[[x]])$ is isomorphic to the ring of formal Laurent series:

$$K((x)) := \left\{a_{-n}x^{-n} + a_{-n+1}x^{-n+1} + a_{-n+2}x^{-n+2} + \cdots : a_i \in K \text{ for all } i \geq -n\right\}.$$

Proof. Given power series $f(x) = \sum_k a_k x^k$ and $g(x) = \sum_\ell b_\ell x^\ell$ recall that the coefficient of x^m in $f(x)g(x)$ is given by $\sum_{k+\ell=m} a_k b_\ell$. I claim that $\mathrm{ord}(fg) = \mathrm{ord}(f) + \mathrm{ord}(g)$. Indeed, if $m < \mathrm{ord}(f)+\mathrm{ord}(g)$ then $k+\ell = m$ implies that either $k < \mathrm{ord}(f)$ or $\ell < \mathrm{ord}(g)$ (otherwise we have $k+\ell \geq \mathrm{ord}(f)+\mathrm{ord}(g) > m$). Thus every term in the sum $\sum_{k+\ell=m} a_k b_\ell$ is zero. However, if $m = \mathrm{ord}(f)+\mathrm{ord}(g)$ then the coefficient of x^m in $f(x)g(x)$ is $\sum_{k+\ell=m} a_k b_\ell = a_{\mathrm{ord}(f)} b_{\mathrm{ord}(g)} \neq 0$ because $a_{\mathrm{ord}(f)} \neq 0$ and $b_{\mathrm{ord}(g)} \neq 0$ by assumption (and K is a domain). We conclude that $\mathrm{ord}(fg) = \mathrm{ord}(f) + \mathrm{ord}(g)$.

For part (a), assume that $f, g \in K[[x]]$ are nonzero. This implies that $\mathrm{ord}(f), \mathrm{ord}(g) < \infty$ and hence $\mathrm{ord}(fg) = \mathrm{ord}(f) + \mathrm{ord}(g) < \infty$. We conclude that $K[[x]]$ is a domain.

For part (b), consider $f(x) = \sum_k a_k x^k$ and $g(x) = \sum_\ell b_\ell x^\ell$ with $g \neq 0$ (i.e. with $\mathrm{ord}(g) < \infty$).We want to prove that there exist $q, r \in K[[x]]$ with $f = qg + r$ and either $r = 0$ or $\mathrm{ord}(r) < \mathrm{ord}(g)$. Indeed, if $\mathrm{ord}(f) < \mathrm{ord}(g)$ then we can simply take $q(x) = 0$ and $r(x) = f(x)$. If $\mathrm{ord}(f) \geq \mathrm{ord}(g)$ then we can perform "long division" as follows. Let b be the lowest coefficient of $g(x)$. Then let $f_1 = f$ and for all $n \geq 1$ such that $f_n \neq 0$ define

$$f_{n+1}(x) := f_n(x) - \frac{a_n}{b}\, x^{\mathrm{ord}(f_n)-\mathrm{ord}(g)} g(x).$$

where a_n is the lowest coefficient of $f_n(x)$. By construction we have $\mathrm{ord}(g) \leq \mathrm{ord}(f_1) < \mathrm{ord}(f_2) < \mathrm{ord}(f_3) < \cdots$ so this is always defined. If the algorithm terminates with $f_N = 0$ then we set $a_n = 0$ for all $n \geq N$, otherwise we let the algorithm run forever (i.e. use induction). In the end we obtain a formal power series

$$q(x) := \sum_{n\geq 1} \frac{a_n}{b}\, x^{\mathrm{ord}(f_n)-\mathrm{ord}(g)}$$

with the property that $f(x) = q(x)g(x)$ (the remainder is always zero!). This proves that $K[[x]]$ is Euclidean.

[Probably a proof by example would have been better, but I didn't feel like typesetting an infinite long division in LaTeX. I encourage you to compute an example yourself.]

In the proof of (b) note that we actually showed that given two power series $f, g \in K[[x]]$ we have $g|f$ if and only if $\mathrm{ord}(g) \leq \mathrm{ord}(f)$. For part (c), note that $g \in K[[x]]$ is a unit if and only if g divides 1. By the above remark this happens if and only if $\mathrm{ord}(g) \leq \mathrm{ord}(1) = 0$, i.e., if and only if $\mathrm{ord}(g) = 0$. Finally, note that $\mathrm{ord}(g) = 0$ if and only if g has nonzero constant term.

For part (d), note that the set of nonunits of $K[[x]]$ are just the power series with zero constant term, i.e., the power series divisible by x:

$$(x) := \{xf(x) : f(x) \in K[[x]]\}.$$

Since this is an ideal we conclude that $K[[x]]$ is a local ring.

For part (e), we say that $f(x) = \sum_k a_k x^k$ is a formal Laurent series if there exists a minimum $r \in \mathbb{Z}$ (possibly negative) such that $a_r \neq 0$. In this case we define $\mathrm{ord}(f) = r$. Let $K((x))$ denote the ring of formal Laurent series with addition and multiplication defined just as for power series. Then $K[[x]] \subseteq K((x))$ is the subring or Laurent series with nonnegative order.

I claim that $K((x))$ is a field. Indeed, given **any** two Laurent series $f, g \in K((x))$ with $g \neq 0$, the long division process defined above can be used to obtain $q(x) \in K((x))$ such that $f(x) = q(x)g(x)$. We we do not require $\text{ord}(g) \leq \text{ord}(f)$. In fact, because $\text{ord}(q) = \text{ord}(f) - \text{ord}(g)$ we will have $q \in K[[x]]$ if and only if $\text{ord}(g) \leq \text{ord}(f)$. If $f(x) = 1$ then we obtain $q(x) = g(x)^{-1}$.

Since $K((x))$ is a field containing $K[[x]]$ we can identify $\text{Frac}(K[[x]])$ with the subfield of $K((x))$ consisting of elements of the form $f(x)g(x)^{-1}$ for $f, g \in K[[x]]$ with $g \neq 0$. But note that **every** Laurent series $f(x) \in K((x))$ has this form. Indeed, if $\text{ord}(f) \geq 0$ then $f(x) = f(x)(1)^{-1} \in \text{Frac}(K[[x]])$ and if $\text{ord}(f) < 0$ then

$$f(x) = (x^{-\text{ord}(f)} f(x))(x^{-\text{ord}(f)})^{-1} \in \text{Frac}(K[[x]])$$

because $x^{-\text{ord}(f)} f(x)$ and $x^{-\text{ord}(f)}$ are in $K[[x]]$. We conclude that

$$\text{Frac}(K[[x]]) = K((x)).$$

$\square$

[As you may know, any function $f : \mathbb{C} \to \mathbb{C}$ holomorphic in an annulus has a convergent Laurent series expansion there. This makes complex analysis a very algebraic subject.]

Problem 3. (Partial Fraction Expansion) To what extent can we "un-add" fractions? Let R be a PID. Consider $a, b \in R$ with $b = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ where $p_1, \ldots, p_k$ are distinct primes and $e_1, \ldots, e_k \geq 1$.

(a) Prove that there exist $a_1, \ldots, a_k \in R$ such that

$$\frac{a}{b} = \frac{a_1}{p_1^{e_1}} + \frac{a_2}{p_2^{e_2}} + \cdots + \frac{a_k}{p_k^{e_1}}.$$

[Hint: First prove it when $b = pq$ with p, q coprime. Use Bézout.]

Now assume that R is a Euclidean domain with norm function $N : R - \{0\} \to \mathbb{N}$.

(b) Prove that there exist $c, r_{ij} \in R$ such that

$$\frac{a}{b} = c + \sum_{i=1}^{k} \sum_{j=1}^{e_i} \frac{r_{ij}}{p_i^j},$$

where for all i, j we have either $r_{ij} = 0$ or $N(r_{ij}) < N(p_i)$. [Hint: If p is prime, prove that we can write $\frac{a}{p^e}$ as $\frac{q}{p^{e-1}} + \frac{r}{p^e}$ where either $r = 0$ or $N(r) < N(p)$. Then use (a).]

Now suppose that the norm function satisfies $N(a) \leq N(ab)$ and $N(a-b) \leq \max\{N(a), N(b)\}$ for all $a, b \in R - \{0\}$.

(c) Prove that the partial fraction expansion from part (b) is **unique**. [Hint: Suppose we have two expansions

$$c + \sum_{i=1}^{k} \sum_{j=1}^{e_i} \frac{r_{ij}}{p_i^j} = \frac{a}{b} = c' + \sum_{i=1}^{k} \sum_{j=1}^{e_i} \frac{r'_{ij}}{p_i^j}.$$

Then we get a partial fraction expansion of zero:

$$\frac{0}{b} = \frac{a - a}{b} = (c - c') + \sum_{i=1}^{k} \sum_{j=1}^{e_i} \frac{(r_{ij} - r'_{ij})}{p_i^j}.$$

For all i, j define $\hat{b}_{ij} := b/p_i^j$, so that

$$b(c' - c) = \sum_{i=1}^{k} \sum_{j=1}^{e_i} (r_{ij} - r'_{ij})\hat{b}_{ij}.$$

Suppose for contradiction that there exist i, j such that $r_{ij} \neq r'_{ij}$ and let j be maximal with this property. Use the last equation above to show that p_i divides $(r_{ij} - r'_{ij})$ and hence

$$N(p_i) \leq N(r_{ij} - r'_{ij}) \leq \max\{N(r_{ij}), N(r'_{ij})\} < N(p_i).$$

Contradiction.]

(d) If K is a field and $R = K[x]$ then the norm function $N(f) = \deg(f)$ satisfies the hypotheses of part (c) so the expansion is unique. **Compute** the unique expansion of

$$\frac{x^5 + x + 1}{(x+1)^2(x^2+1)} \in \mathbb{R}(x).$$

(e) If $R = \mathbb{Z}$ then the norm function $N(a) = |a|$ does **not** satisfy $|a - b| \leq \max\{|a|, |b|\}$. However, if we require remainders r, r' to be nonnegative then it is true that $|r - r'| \leq \max\{|r|, |r'|\}$ and the proof of uniqueness in (c) still goes through. **Compute** the unique expansion of $\frac{77}{12} \in \mathbb{Q}$ with nonnegative parameters $r_{ij} \geq 0$.

Proof. Consider $a, b \in R$ with $b = p_1^{e_1} \cdots p_k^{e_k}$ where $p_1, \ldots, p_k$ are distinct primes and $e_1, \ldots, e_k \geq 1$. For part (a), note that since R is a PID we must have $(p_1^{e_1}, p_2^{e_2} \cdots p_k^{e_k}) = (d)$ where d is the greatest common divisor. Since $p_1^{e_1}$ and $p_2^{e_2} \cdots p_k^{e_k}$ are coprime this implies that $d = 1$, and hence there exist $c_1, c_2 \in R$ such that

$$1 = c_1 p_2^{e_2} \cdots p_k^{e_k} + c_2 p_1^{e_1}.$$

Multiplying both sides by $\frac{a}{b}$ gives

$$\frac{a}{b} = \frac{ac_1}{p_1^{e_1}} + \frac{ac_2}{p_2^{e_2} \cdots p_k^{e_k}}.$$

Now the result follows by induction.

For part (b), let R be Euclidean and suppose that we have written

$$\frac{a}{b} = \frac{a_1}{p_1^{e_1}} + \cdots + \frac{a_k}{p_k^{e_k}}.$$

Now consider any $\frac{a}{p^e}$ with $a, p \in R$ and p prime. We can divide a by p to obtain $q, r \in R$ such that $a = pq + r$ and either $r = 0$ or $N(r) < N(p)$. In other words, we have

$$\frac{a}{p^e} = \frac{qp + r}{p^e} = \frac{q}{p^{e-1}} + \frac{r}{p^e}$$

where either $r = 0$ or $N(r) < N(p)$. By induction we obtain

$$\frac{a}{p^e} = \frac{q}{p^0} + \frac{r_1}{p^1} + \frac{r_2}{p^2} + \cdots + \frac{r_e}{p^e},$$

where for all i we have $r_i = 0$ or $N(r_i) < N(p)$. Then, combining these expressions for each summand $\frac{a_i}{p_i^{e_i}}$ of $\frac{a}{b}$ gives

$$\frac{a}{b} = c + \sum_{i=1}^{k} \sum_{j=1}^{e_i} \frac{r_{ij}}{p_i^j},$$

where for all i, j we have $r_{ij} = 0$ or $N(r_{ij}) < N(p_i)$. This is called a "partial fraction expansion" of $\frac{a}{b}$.

For part (c), suppose that the Euclidean norm satisfies $N(a) \leq N(ab)$ and $N(a-b) \leq \max\{N(a), N(b)\}$ for all $a, b \in R - \{0\}$, and suppose we have two partial fraction expansions

$$c + \sum_{i=1}^{k} \sum_{j=1}^{e_i} \frac{r_{ij}}{p_i^j} = \frac{a}{b} = c' + \sum_{i=1}^{k} \sum_{j=1}^{e_i} \frac{r'_{ij}}{p_i^j}.$$

I claim that $r_{ij} = r'_{ij}$ for all i, j (and hence also $c = c'$). To see this, we subtract the expansions:

$$0 = (c' - c) + \sum_{i=1}^{k} \sum_{j=1}^{e_i} \frac{(r_{ij} - r'_{ij})}{p_i^j}$$

Then we multiply both sides by b to get

$$b(c - c') = \sum_{i=1}^{k} \sum_{j=1}^{e_i} (r_{ij} - r'_{ij}) \hat{b}_{ij},$$

where $\hat{b}_{ij} := b/p_i^j \in R$. Now assume for contradiction that we have $r_{mn} \neq r'_{mn}$ for some $m, n \geq 1$ and let n be **maximal** with this property. That is, suppose that we also have $r_{mj} = r'_{mj}$ for all $j > n$. In this case, note that $p_m^{e_m - n}$ divides $\hat{b}_{ij}$ for every nonzero term in the sum, thus since R is a domain we can cancel it to get

$$b'(c - c') = \sum_{i=1}^{k} \sum_{j=1}^{e_i} (r_{ij} - r'_{ij}) \hat{b}'_{ij}, \tag{4}$$

where

$$b' = p_1^{e_1} \cdots p_m^{n} \cdots p_k^{e_k} \quad \text{and} \quad \hat{b}'_{ij} = \begin{cases} p_1^{e_1} \cdots p_i^{e_i - j} \cdots p_m^{n} \cdots p_k^{e_k} & i < m \\ p_1^{e_1} \cdots p_i^{e_i - j} \cdots p_m^{n} \cdots p_k^{e_k} & i > m \\ p_1^{e_1} \cdots p_m^{n-j} \cdots p_k^{e_k} & i = m, j \leq n \\ 0 & i = m, j > n \end{cases}$$

Finally, note that p_m divides $(r_{mn} - r'_{mn})\hat{b}'_{mn}$ because p_m divides every other term of the sum (4). Since p_m is prime, Euclid says that $p_m | (r_{mn} - r'_{mn})$ or $p_m | \hat{b}'_{mn}$. But by definition we know that p_m does **not** divide $\hat{b}'_{mn}$. We conclude that p_m divides $r_{mn} - r'_{mn}$ and then the assumed properties of the norm imply that

$$N(p_m) \leq N(r_{mn} - r'_{mn}) \leq \max\left\{N(r_{mn}), N(r'_{mn})\right\} < N(p_m).$$

Contradiction.

For parts (d) and (e) I will naively follow the steps of the proof. I will not use any tricks like differentiation. (You can get the solution faster with tricks.) For part (d) we first look for polynomials $f(x)$ and $g(x)$ such that

$$1 = f(x)(x+1)^2 + g(x)(x^2+1).$$

For this we consider the set of triples $f, g, h \in \mathbb{R}[x]$ with $f(x)(x+1)^2 + g(x)(x^2+1) = h(x)$ and apply row reduction:

$f(x)$	$g(x)$	$h(x)$
1	0	$(x+1)^2$
0	1	x^2+1
1	-1	$2x$
$-x/2$	$1 + x/2$	1

We conclude that $(-x/2)(x+1)^2 + (1+x/2)(x^2+1) = 1$ and hence

$$\begin{aligned}\frac{1}{(x+1)^2(x^2+1)} &= \frac{(-x/2)(x+1)^2 + (1+x/2)(x^2+1)}{(x+1)^2(x^2+1)}\\ &= \frac{-x/2}{x^2+1} + \frac{1+x/2}{(x+1)^2}.\end{aligned}$$

Multiplying both sides by x^5+x+1 gives

$$\frac{x^5+x+1}{(x+1)^2(x^2+1)} = \frac{-\frac{1}{2}(x^6+x^2+x)}{x^2+1} + \frac{\frac{1}{2}(x^6+2x^5+x^2+3x+2)}{(x+1)^2}.$$

Now we deal with both of the summands separately. First we divide $-\frac{1}{2}(x^6+x^2+x)$ by x^2+1 to get

$$-\frac{1}{2}(x^6+x^2+x) = -\frac{1}{2}(x^4-x^2+2)(x^2+1) - \frac{1}{2}(x-2),$$

hence

$$\frac{-\frac{1}{2}(x^6+x^2+x)}{x^2+1} = -\frac{1}{2}(x^4-x^2+2) + \frac{-\frac{1}{2}(x-2)}{(x^2+1)}.$$

Next we divide $\frac{1}{2}(x^6+2x^5+x^2+3x+2)$ by $(x+1)$ to get

$$\frac{1}{2}(x^6+2x^5+x^2+3x+2) = \frac{1}{2}(x^5+x^4-x^3+x^2+3)(x+1) - \frac{1}{2},$$

hence

$$\frac{\frac{1}{2}(x^6+2x^5+x^2+3x+2)}{(x+1)^2} = \frac{\frac{1}{2}(x^5+x^4-x^3+x^2+3)}{(x+1)} + \frac{-1/2}{(x+1)^2}.$$

Finally, we divide $\frac{1}{2}(x^5+x^4-x^3+x^2+3)$ by $(x+1)$ to get

$$\frac{1}{2}(x^5+x^4-x^3+x^2+3) = \frac{1}{2}(x^4-x^2+2x-2)(x+1) + 5,$$

hence

$$\frac{\frac{1}{2}(x^5+x^4-x^3+x^2+3)}{(x+1)} = \frac{1}{2}(x^4-x^2+2x-2) + \frac{5/2}{(x+1)}.$$

Putting everything together gives

$$\boxed{\frac{x^5+x+1}{(x+1)^2(x^2+1)} = (x-2) + \frac{5/2}{(x+1)} + \frac{-1/2}{(x+1)^2} + \frac{-(x-2)/2}{(x^2+1)}.}$$

[By doing everything out longhand I meant to show that it is possible, not that it is easy.]

For part (e) we first factor $12 = 3\cdot 4$ with $3, 4$ coprime. Now we look for $x, y \in \mathbb{Z}$ with $3x+4y=1$. This can be done by inspection:

$$1 = 3(-1) + 4\cdot 1$$

[If it couldn't be done by inspection we would use the Euclidean algorithm.] Dividing by 12 gives

$$\frac{1}{12} = \frac{3(-1)+4\cdot 1}{3\cdot 4} = \frac{-1}{4} + \frac{1}{3},$$

and then multiplying by 77 gives

$$\frac{77}{12} = \frac{-77}{4} + \frac{77}{3}.$$

Now we deal with both of the summands separately. First we divide 77 by 3 to get

$$\begin{aligned} 77 &= 3 \cdot 25 + 2 \\ \frac{77}{3} &= 25 + \frac{2}{3}. \end{aligned}$$

Then we divide -77 by 2 to get

$$\begin{aligned} -77 &= 2(-39) + 1 \\ \frac{-77}{4} &= \frac{-39}{2} + \frac{1}{4}. \end{aligned}$$

Finally, we divide -39 by 2 to get

$$\begin{aligned} -39 &= 2(-20) + 1 \\ \frac{-39}{2} &= -20 + \frac{1}{2}. \end{aligned}$$

Putting everything together gives

$$\boxed{\frac{77}{12} = 5 + \frac{1}{2} + \frac{1}{4} + \frac{2}{3}.}$$

This result is unique as long as we use positive remainders. □

[Why did I ask you to do this? Because I always wondered about partial fractions. They appear in Calculus to show us that all rational functions over $\mathbb{R}$ can be integrated in elementary terms. For example:

$$\int \frac{x^5 + x + 1}{(x+1)^2(x^2+1)}\, dx = \frac{1}{2}x^2 - 2x + \frac{5}{2}\ln(x+1) - \frac{1}{2(x+1)} - \frac{1}{4}\ln(x^2+1) + \arctan(x).$$

But then partial fractions mysteriously disappear from the curriculum. Now at least we know why.]