Math 661 **Fall 2013**
Homework 4 **Drew Armstrong**

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1. Let K be a field and let V be a K-module. A composition series of length n is a chain of submodules

$$0 = V_0 < V_1 < \cdots < V_n = V$$

such that each quotient V_{i+1}/V_i is a simple K-module (has no nontrivial submodules). Recall that if V is finitely generated then every independent generating set has the same size (by Steinitz Exchange), called the dimension $\dim(V)$. Prove that if V has a composition series then V is finitely generated, and furthermore every composition series has the same length $n = \dim(V)$. [Do not quote the Jordan-Hölder Theorem for modules. Hint: Prove by induction that $\dim V_k = k$ for $k = 0, 1, \ldots, n$.]

[Here we are proving the Jordan-Hölder Theorem for R-modules in the special case that R is a field. The proof for general R would require a different technique.]

Proof. We will prove by induction on k that V_k is finitely generated with $\dim V_k = k$. It will follow that V_n is finitely generated with $\dim V_n = n$ as desired.

Now assume for induction that V_k is finitely generated with $\dim V_k = k$, and choose a basis

$$\alpha_1, \alpha_2, \ldots, \alpha_k \in V_k.$$

By assumption the module V_{k+1}/V_k is simple, which means that it is nonzero and has no nontrivial submodules. Let $\beta + V_k$ be a nonzero coset with $\beta \in V_{k+1}$ (the fact that the coset is nonzero means that $\beta \notin V_k$). Then the nonzero submodule $K(\beta + V_k)$ of the simple module V_{k+1}/V_k must be equal to V_{k+1}/V_k. We claim that

$$\beta, \alpha_1, \alpha_2, \ldots, \alpha_k \in V_{k+1}$$

is a basis for V_{k+1}.

First we show that the set generates V_{k+1}. Consider an arbitrary $x \in V_{k+1}$. Then since $V_{k+1}/V_k = K(\beta + V_k)$ there exists some $c \in K$ such that

$$\begin{aligned} & x + V_k = c(\beta + V_k) \\ \Rightarrow \quad & x + V_k = c\beta + V_k \\ \Rightarrow \quad & x - c\beta \in V_k. \end{aligned}$$

This means that there exist $c_i \in K$ such that $x - c\beta = \sum_i c_i\alpha_i$, and hence $x = c\beta + \sum_i c_i\alpha_i$.

Finally we show that the set is linearly independent. Consider any $c, c_1, c_2, \ldots, c_k \in K$ such that $c\beta + \sum_i c_i\alpha_i = 0$. Then we have

$$0 + V_k = (c\beta - \sum_i c_i\alpha_i) + V_k = c\beta + V_k,$$

hence $c\beta \in V_k$. Since $\beta \notin V_k$ this implies that $c = 0$ and hence $\sum_i c_i\alpha_i = 0$. Since the α_i are linearly independent by assumption this implies that $c_i = 0$ for all i. Therefore $\beta, \alpha_1, \ldots, \alpha_k$ is a basis for V_{k+1} and we conclude that $\dim V_{k+1} = k + 1$ as desired. $\square$

2. Consider the defining action of the symmetric group S_n on the set $\{1, 2, \ldots, n\}$.

(a) Let $\mathrm{Gr}_1(k, n)$ denote the set of subsets of $\{1, 2, \ldots, n\}$ of size k. Show that S_n acts transitively on $\mathrm{Gr}_1(k, n)$.

(b) Show that the stabilizer of any k-subset is isomorphic to $S_k \times S_{n-k}$.

(c) Use the Orbit-Stabilizer Theorem to conclude that

$$|\mathrm{Gr}_1(k,n)| = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

Proof. We define the action of $\pi \in S_n$ on a subset X of $\{1,2,\ldots,n\}$ by

$$\pi(X) := \{\pi(x) : x \in X\}.$$

For part (a) consider two subsets $X, Y \subseteq \{1,2,\ldots,n\}$ such that $|X| = |Y| = k$. This allows us to write

$$\{x_1,\ldots,x_k,x_{k+1},\ldots,x_n\} = \{1,2,\ldots,n\} = \{y_1,\ldots,y_k,y_{k+1},\ldots,y_n\},$$

where $X = \{x_1,\ldots,x_k\}$ and $Y = \{y_1,\ldots,y_k\}$. Then permutation $\pi \in S_n$ defined by $\pi(x_i) := y_i$ for all i clearly satisfies $\pi(X) = Y$.

For part (b), consider the subset X from above, let $G \leq S_n$ be the subgroup that fixes the elements of $\{1,\ldots,n\} - X$ and let $H \leq S_n$ be the subgroup that fixes the elements of X. Clearly we have $G \approx S_k$ and $H \approx S_{n-k}$. Furthermore it is easy to see that $G \cap H = 1$ and $gh = hg$ for all $g \in G, h \in H$. We conclude that $GH \approx G \times H$ is a direct product. Now consider any $\pi \in S_n$ such that $\pi(X) = X$. Define $h_\pi \in S_n$ by setting $h_\pi(x) = \pi(x)$ for $x \in X$ and $h_\pi(x) = x$ for $x \in \{1,\ldots,n\} - X$. Notice that $h_\pi \in H$. Similarly we define $g_\pi \in G$ to agree with π on $\{1,\ldots,n\} - X$ and to fix every element of X. We conclude that $\pi = g_\pi h_\pi$ is in GH. Conversely, it is easy to see that any element of GH stabilizes X. We conclude that the stabilizer group of X is $GH \approx S_k \times S_{n-k}$.

Now for part (c). By Orbit-Stabilizer we have a bijection between the set $\mathrm{Gr}_1(k,n)$ of k-subsets of $\{1,\ldots,n\}$ and the coset space S_n/GH. Since everything is finite this implies

$$\begin{aligned}
|\mathrm{Gr}_1(k,n)| &= \frac{|S_n|}{|S_k \times S_{n-k}|} \\
&= \frac{|S_n|}{|S_k| \cdot |S_{n-k}|} \\
&= \frac{n!}{k!\,(n-k)!} \\
&= \binom{n}{k}.
\end{aligned}$$

□

[Remark: The notation $\mathrm{Gr}_1(k,n)$ is nonstandard and is meant to be suggestive.]

For the following problems we define the standard **q-integer**, **q-factorial** and **q-binomial**:

$$\begin{aligned}
[n]_q &= 1 + q + q^2 + \cdots + q^{n-1}, \\
[n]_q! &= [n]_q[n-1]_q \cdots [2]_q[1]_q, \\
\begin{bmatrix} n \\ k \end{bmatrix}_q &= \frac{[n]_q!}{[k]_q!\,[n-k]_q!}.
\end{aligned}$$

3. Let K be a field and consider the defining action of $GL(n,K)$ on K^n.

(a) Let $\mathrm{Gr}_K(k,n)$ denote the set of k-dimensional subspaces of K^n (called the Grassmannian). Show that $GL(n,K)$ acts transitively on $\mathrm{Gr}_K(k,n)$.

(b) Show that the stabilizer of any k-subspace is isomorphic to

$$(GL(k,K)\times GL(n-k,K))\ltimes \mathrm{Mat}_{k,n-k}(K),$$

where $\mathrm{Mat}_{k,n-k}(K)$ is the **additive** group of $k\times(n-k)$ matrices. [Hint: Show that the stabilizer is isomorphic to the group of block matrices

$$\left(\begin{array}{c|c} A & C \\ \hline 0 & B \end{array}\right)$$

where $A\in GL(k,K)$, $B\in GL(n-k,K)$ and $C\in \mathrm{Mat}_{k,n-k}(K)$.]

(c) In the case $K=\mathbb{F}_q$ we will write $\mathrm{Gr}_K(k,n)=\mathrm{Gr}_q(k,n)$. Use part (b) and the Orbit-Stabilizer Theorem to show that

$$|\mathrm{Gr}_q(k,n)|=\begin{bmatrix} n\\ k\end{bmatrix}_q.$$

[Hint: Show that $|GL(n,q)|=q^{\binom{n}{2}}(q-1)^n[n]_q!$.]

Proof. We define the action of $g\in GL(n,K)$ on a subset $X\subseteq K^n$ by

$$g(X):=\{gx:x\in X\}.$$

For part (a), let U and V be any two k-dimensional subspaces of K^n and choose bases $\mathbf{u}_1,\ldots,\mathbf{u}_k\in U$ and $\mathbf{v}_1,\ldots,\mathbf{v}_k\in V$. By Steinitz we can extend each of these to a basis of K^n to obtain $\mathbf{u}=(\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_n)$ and $\mathbf{v}=(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n)$. We define a function $\varphi:K^n\to K^n$ by sending $\varphi(\mathbf{u}_i):=\mathbf{v}_i$ and extending linearly. If we write φ as a matrix in standard coordinates we obtain the change of basis matrix

$$[\varphi]=[\mathbf{u}\to\mathbf{v}]:=\begin{pmatrix}[\mathbf{u}_1]_{\mathbf{v}} & [\mathbf{u}_2]_{\mathbf{v}} & \cdots & [\mathbf{u}_n]_{\mathbf{v}}\end{pmatrix}\in GL(n,K),$$

where $[\mathbf{u}_j]_{\mathbf{v}}$ is the column vector of $\mathbf{u}_j$ expressed in $\mathbf{v}$ coordinates. Finally, it is straightforward to show that $[\mathbf{u}\to\mathbf{v}](U)=V$.

For part (b) consider the k-dimensional subspace $U\le K^n$ from above. We will write all vectors and matrices in terms of the basis $\mathbf{u}$. Now let P be the set of block matrices of the form

$$P:=\left\{\left(\begin{array}{c|c} A & C \\ \hline 0 & B \end{array}\right): A\in GL(k,n), B\in GL(n-k,K), C\in \mathrm{Mat}_{k,n-k}(K)\right\}.$$

Note that this is a group with

$$\left(\begin{array}{c|c} A & C \\ \hline 0 & B \end{array}\right)\left(\begin{array}{c|c} A' & C' \\ \hline 0 & B' \end{array}\right)=\left(\begin{array}{c|c} AA' & AC'+CB' \\ \hline 0 & BB' \end{array}\right).$$

It follows that inverses are given by

$$\left(\begin{array}{c|c} A & C \\ \hline 0 & B \end{array}\right)^{-1}=\left(\begin{array}{c|c} A^{-1} & -A^{-1}CB^{-1} \\ \hline 0 & B^{-1} \end{array}\right).$$

We will also partition vectors $x\in K^n$ by writing

$$x=\left(\begin{array}{c} x_1\\ \vdots \\ x_k \\ \hline x_{k+1} \\ \vdots \\ x_n\end{array}\right)=\left(\begin{array}{c} x_U \\ \hline x_U' \end{array}\right).$$

Note that we have $x \in U$ if and only if $x'_U = 0$. Then we observe that the group P stabilizes the subspace U because

$$\left(\begin{array}{c|c} A & C \\ \hline 0 & B \end{array}\right)\left(\begin{array}{c} x_U \\ \hline 0 \end{array}\right) = \left(\begin{array}{c} Ax_U \\ \hline 0 \end{array}\right)$$

Conversely, note that for a general element $g \in GL(n, K)$ and $x \in U$ we have

$$gx = \left(\begin{array}{c|c} A & C \\ \hline D & B \end{array}\right)\left(\begin{array}{c} x_U \\ \hline 0 \end{array}\right) = \left(\begin{array}{c} Ax_U \\ \hline Dx_U \end{array}\right).$$

If g stabilizes U — that is, if $gx \in U$ for all $x \in U$ — then we must have $D = 0$ and hence $g \in P$. We conclude that P is the stabilizer of U.

Now we will prove that $P \approx (GL(k, K) \times GL(n-k, K)) \ltimes \mathrm{Mat}_{k,n-k}(K)$. To do this we will let G denote the subgroup of P where $C = 0$ and $B = I$, let H denote the subgroup of P where $C = 0$ and $A = I$, and let M denote the subgroup of P where $A = I$ and $C = I$. Clearly we have $G \approx GL(k, K)$ and $H \approx GL(n-k, K)$. It is easy to see that GH is a subgroup of P isomorphic to $G \times H$ because $G \cap H = 1$ and the groups commute:

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & I \end{array}\right)\left(\begin{array}{c|c} I & 0 \\ \hline 0 & B \end{array}\right) = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right) = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & B \end{array}\right)\left(\begin{array}{c|c} A & 0 \\ \hline 0 & I \end{array}\right)$$

Note that we also have $M \approx \mathrm{Mat}_{k,n-k}(K)$ as **additive** groups because

$$\left(\begin{array}{c|c} I & C \\ \hline 0 & I \end{array}\right)\left(\begin{array}{c|c} I & C' \\ \hline 0 & I \end{array}\right) = \left(\begin{array}{c|c} I & C+C' \\ \hline 0 & I \end{array}\right)$$

Next observe that $(G \times H) \cap M = 1$ and that $(G \times H)M = P$ because

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)\left(\begin{array}{c|c} I & A^{-1}C \\ \hline 0 & I \end{array}\right) = \left(\begin{array}{c|c} A & C \\ \hline 0 & B \end{array}\right).$$

It remains only to show that M is normal in P. Indeed, we have

$$\begin{aligned}
&\left(\begin{array}{c|c} A & C \\ \hline 0 & B \end{array}\right)\left(\begin{array}{c|c} I & D \\ \hline 0 & I \end{array}\right)\left(\begin{array}{c|c} A & C \\ \hline 0 & B \end{array}\right)^{-1} \\
&= \left(\begin{array}{c|c} A & AD+C \\ \hline 0 & B \end{array}\right)\left(\begin{array}{c|c} A^{-1} & -A^{-1}CB^{-1} \\ \hline 0 & B^{-1} \end{array}\right) \\
&= \left(\begin{array}{c|c} AA^{-1} & -AA^{-1}CB^{-1} + (AD+C)B^{-1} \\ \hline 0 & BB^{-1} \end{array}\right) \\
&= \left(\begin{array}{c|c} I & -CB^{-1} + ADB^{-1} + CB^{-1} \\ \hline 0 & I \end{array}\right) \\
&= \left(\begin{array}{c|c} I & ADB^{-1} \\ \hline 0 & I \end{array}\right).
\end{aligned}$$

We conclude that $P = (G \times H) \ltimes M$ as desired, and the conjugation action of $G \times H$ on M is our **favorite** action of $GL(k, K) \times GL(n-k, K)$ on $\mathrm{Mat}_{n-k,k}(K)$, namely

$$(A, B) \bullet D := ADB^{-1}.$$

Now for part (c). By the Orbit-Stabilizer Theorem we have identified the Grassmannian $\mathrm{Gr}_K(k, n)$ with the coset space $GL(n, K)/P$. Now assume that we are working over the finite field $K = \mathbb{F}_q$. We proved in class that $|GL(n, q)| = q^{\binom{n}{2}}(q-1)^n[n]_q!$ and we can see that $|\mathrm{Mat}_{k,n-k}(q)| = q^{k(n-k)}$ because the entries are unrestricted. Note that we also have $|P| = |(G \times H) \ltimes M| = |G \times H||M| = |G||H||M|$. Putting everything together gives

$$
\begin{aligned}
|\mathrm{Gr}_q(n,k)| &= \frac{|GL(n,q)|}{|GL(k,q)| \cdot |GL(n-k,q)| \cdot |\mathrm{Mat}_{k,n-k}(q)|} \\
&= \frac{q^{n(n-1)/2}(q-1)^n [n]_q!}{q^{k(k-1)/2}(q-1)^k [k]_q! \cdot q^{(n-k)(n-k-1)/2}(q-1)^{n-k}[n-k]_q! \cdot q^{k(n-k)}} \\
&= \frac{[n]_q!}{[k]_q!\,[n-k]_q!}.
\end{aligned}
$$

□

[Remark: P is for Parabolic. Subgroups of block upper triangular matrices are called "parabolic" for some forgotten reason. A Borel subgroup $B \leq GL(n,K)$ is an example of a parabolic subgroup. Parabolic subgroups of $GL(n,K)$ are precisely those subgroups containing B.]

4. Let K be a field and consider the defining action of $GL(n,K)$ on K^n.

(a) Let $\mathrm{Flag}_K(n)$ denote the set of complete flags (called the complete flag variety)

$$0 = V_0 < V_1 < \cdots < V_n = K^n.$$

Show that $GL(n,K)$ acts transitively on $\mathrm{Flag}_K(n)$.

(b) Let $B \leq GL(n,K)$ denote the stabilizer of any complete flag. Show that B is isomorphic to the group of upper triangular matrices. (Remark: B is for Armand Borel.)

(c) In the case $K = \mathbb{F}_q$ we will write $\mathrm{Flag}_K(n) = \mathrm{Flag}_q(n)$. Use part (b) and the Orbit-Stabilizer Theorem to show that

$$|\mathrm{Flag}_q(n)| = [n]_q!.$$

[Hint: Show that $|B| = q^{\binom{n}{2}}(q-1)^n$.]

Proof. Given a complete flag

$$0 = V_0 < V_1 < \cdots < V_n = K^n,$$

we know from Problem 1 that $\dim V_k = k$ for all k. By applying Steinitz repeatedly, we can choose a basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ for K^n such that $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is a basis for V_k. For part (a) consider two complete flags with their corresponding bases. Then the change of basis matrix sends one flag to the other. (See the solution to Problem 2(a) for details.)

For part (b), consider the complete flag above with basis $\mathbf{v} = \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in K^n$. We will express all vectors and matrices in $\mathbf{v}$ coordinates. Suppose that the matrix $A \in GL(n,K)$ stabilizes the flag, i.e., suppose that $A(V_k) = V_k$ for all k. In particular this means that $A\mathbf{v}_k \in V_k$. But note that $\mathbf{v}_k$ written in $\mathbf{v}$ coordinates is the column vector with 1 in the k-th position and zeroes elsewhere, hence $A\mathbf{v}_k$ is just the k-th column of A. Since $A\mathbf{v}_k \in V_k$ we conclude that the k-th column of A has zeroes below the k-th position. Hence A is upper triangular. Conversely, given any upper triangular $A \in GL(n,K)$ it is easy to see that A stabilizes V_k for all k. We conclude that the stabilizer of the complete flag is the "Borel subgroup" $B \leq GL(n,K)$ of upper triangular matrices.

By the Orbit-Stabilizer Theorem we now have a correspondence between the flag variety $\mathrm{Flag}_K(n)$ and the coset space $GL(n,K)/B$. For part (c) we consider the finite field $K = \mathbb{F}_q$. We know from before that $|GL(n,q)| = q^{\binom{n}{2}}(q-1)^n[n]_q!$. Now observe that $|B| = q^{\binom{n}{2}}(q-1)^n$. Indeed, the diagonal entries can be any nonzero field elements so there are $(q-1)^n$ ways to choose them. The above-diagonal entries are unrestricted. Since the number of above-diagonal entries is

$$1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2} = \binom{n}{2},$$

there are $q^{\binom{n}{2}}$ ways to choose them. Finally, we have

$$\begin{aligned} |\mathrm{Flag}_q(n)| &= \frac{|GL(n,q)|}{|B|} \\ &= \frac{q^{\binom{n}{2}}(q-1)^n[n]_q!}{q^{\binom{n}{2}}(q-1)^n} \\ &= [n]_q!. \end{aligned}$$

□

[Remark: In general, a *Borel subgroup* is a maximal closed solvable connected subgroup of a linear algebraic group. The prototypical example is the subgroup of upper triangular matrices in $GL(n, K)$.]

[Remark: Let $\mathbb{F}_1$ denote the field with one element. No such thing exists, but don't worry. There is a useful heuristic that says that the set $\{1, 2, \ldots, n\}$ is like an n-dimensional vector space V over $\mathbb{F}_1$, a subset of size k is like a k-dimensional subspace, and a permutation of $\{1, 2, \ldots, n\}$ is like a complete flag of subspaces. If you perform the linear algebra over $\mathbb{F}_q$ and then take $q \to 1$ in any relevant formulas, you should get reasonable answers. To my knowledge, no one has given a satisfactory explanation of this.]