Math 661 **Fall 2013**
Homework 3 **Drew Armstrong**

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1. Given a group, define its center (Zentrum):

$$Z(G) := \{g \in G : gh = hg \text{ for all } h \in G\}.$$

Note that $Z(G)$ is abelian and $Z(G) \trianglelefteq G$. If $G/Z(G)$ is cyclic, show that G is abelian.

2. Let p be prime and consider a group G of order p^2.

(a) Use the class equation to show that p divides $|Z(G)|$.
(b) Use Problem 1 to show that G must be abelian.
(c) Show that G must be isomorphic to $\mathbb{Z}/p^2$ or $\mathbb{Z}/p \times \mathbb{Z}/p$.

3. Let p be prime and let G be a group of order $2p$.

(a) Prove that $G \approx \langle x \rangle \ltimes \langle y \rangle$ where $\langle x \rangle$ is a cyclic group of order 2 and $\langle y \rangle$ is cyclic group of order p that is normal in G. [Hint: Sylow.]
(b) Note that $xyx = y^i$ for some $i \in \mathbb{Z}$. In this case, show that $y^{i^2-1} = 1$. [Hint: Consider the element x^2yx^2.]
(c) If $y^{i^2-1} = 1$, show that $p|(i-1)$ or $p|(i+1)$. Show that this implies that $xyx = y$ (and hence G is cyclic) or $xyx = y^{-1}$ (and hence G is dihedral).

4. Prove that the alternating group A_4 is not simple. [Hint: Consider permutations of the form $(ij)(k\ell)$. Recall that conjugate permutations have the same cycle type.]

5. If $|G| = 30$, prove that G is not simple. [Hint: Let P and Q be a Sylow 5-subgroup and a Sylow 3-subgroup. Show that at least one of them must be normal in G. (If not, show that there are too many elements of order 5 and order 3.) Conclude that $PQ \lhd G$.]