Math 592 (The Math of Music) **Spring 2015**
Quiz 3 **Drew Armstrong**

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1. Write down the solution of the **undamped** harmonic oscillator $x''(t) + \omega^2 x(t) = 0$ with initial conditions $x(0)$ and $x'(0)$.

$$x(t) = x(0)\cos(\omega t) + \frac{x'(0)}{\omega}\sin(\omega t)$$

2. What is the **frequency** of the damped harmonic oscillator $x''(t) + x'(t) + x(t) = 0$?

$$\frac{\sqrt{3}}{2} \text{ radians per second} \qquad \text{OR} \qquad \frac{\sqrt{3}}{4\pi} \text{ cycles per second}$$

3. For which (real) values of γ will the following system oscillate?

$$x''(t) + \gamma x'(t) + x(t) = 0.$$

The frequency of the general damped oscillator $x''(t) + \gamma x'(t) + \omega^2 x(t) = 0$ is

$$\omega' = \frac{1}{2}\sqrt{4\omega^2 - \gamma^2}.$$

(Note that $\omega' = \omega$ when $\gamma = 0$.) However, if ω' is imaginary (i.e., if $4\omega^2 - \gamma^2 < 0$) then the solution is given by hyperbolic functions and the system doesn't really oscillate. It **will** oscillate if there is not too much damping, i.e., if

$$\begin{aligned} 4\omega^2 - \gamma^2 &> 0 \\ 4\omega^2 &> \gamma^2 \\ 2|\omega| &> |\gamma|. \end{aligned}$$

In our case we have $\omega = 1$ so the system will oscillate when

$$|\gamma| < 2.$$

4. Express $\frac{1}{2}\cos(\omega t) - \frac{\sqrt{3}}{2}\sin(\omega t)$ in the form $r \cdot \cos(\omega t + \varphi)$.

Using the angle sum formula gives

$$r \cdot \cos(\omega t + \varphi) = r\cos\varphi\cos(\omega t) - r\sin\varphi\sin(\omega t).$$

Then letting $r\cos\varphi = \frac{1}{2}$ and $-r\sin\varphi = -\frac{\sqrt{3}}{2}$ gives

$$r = 1 \qquad \text{and} \qquad \varphi = \frac{\pi}{3}$$

We conclude that

$$\frac{1}{2}\cos(\omega t) - \frac{\sqrt{3}}{2}\sin(\omega t) = \cos\left(\omega t + \frac{\pi}{3}\right).$$

5. Compute the exponential of the matrix $\begin{pmatrix} -t & -2t \\ 2t & -t \end{pmatrix}$.

Note that $\left(\begin{smallmatrix} -t & -2t \\ 2t & -t \end{smallmatrix}\right) = \left(\begin{smallmatrix} -t & 0 \\ 0 & -t \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & -2t \\ 2t & 0 \end{smallmatrix}\right)$. Since these two matrices commute (indeed, $\left(\begin{smallmatrix} -t & 0 \\ 0 & -t \end{smallmatrix}\right)$ is just a multiple of the identity matrix so it commutes with everything), we have

$$\begin{aligned}
\exp\begin{pmatrix} -t & -2t \\ 2t & -t \end{pmatrix} &= \exp\left(\begin{pmatrix} -t & 0 \\ 0 & -t \end{pmatrix} + \begin{pmatrix} 0 & -2t \\ 2t & 0 \end{pmatrix}\right) \\
&= \exp\begin{pmatrix} -t & 0 \\ 0 & -t \end{pmatrix} \exp\begin{pmatrix} 0 & -2t \\ 2t & 0 \end{pmatrix} \\
&= \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{pmatrix} \\
&= e^{-t} \begin{pmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{pmatrix}.
\end{aligned}$$