Math 592 **Spring 2015**
Homework 2 **Drew Armstrong**

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1. Beats Again. Show that the phenomenon of beats is independent of "phase shifts". [Hint: Consider the superposition $\sin(f_1 \cdot 2\pi t + \varphi) + \sin(f_2 \cdot 2\pi t + \mu)$.]

Applying the identity $\sin(u) + \sin(v) = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$ gives

$$\sin(f_1 \cdot 2\pi t + \varphi) + \sin(f_2 \cdot 2\pi t + \mu) =$$
$$2 \cdot \sin\left(\frac{f_1+f_2}{2} \cdot 2\pi t + \frac{\varphi+\mu}{2}\right) \cdot \cos\left(\frac{f_1-f_2}{2} \cdot 2\pi t + \frac{\varphi-\mu}{2}\right)$$

This means if we draw the graph of the superposition of two sine waves of frequency f_1 and f_2 (independent of phase shifts), we get an envelope sine wave of frequency $(f_1 - f_2)/2$ containing a sine wave oscillating at frequency $(f_1 + f_2)/2$. The phase shifts $(\varphi - \mu)/2$ and $(\varphi + \mu)/2$ do not change the overall look of the graph, and they do not change the sound: it still sounds like a tone of frequency $(f_1 + f_2)/2$ turning on and off $|f_1 - f_2|$ times per second.

2. Galileo's Theory of Dissonance. Let $a, b, A, B \in \mathbb{R}$. Show that the function

$$x(t) = A\sin(at) + B\cos(bt)$$

is periodic if and only if the number a/b equals a fraction of whole numbers. In this case, what is the period? Galileo believed that this is the reason we prefer ratios of small whole numbers: so that our eardrum is not "kept in perpetual torment". [Hint: If $x(t)$ has period T then so does $x''(t) + b^2x(t)$. Does this put any restriction on T?]

Note that $A\sin(at)$ has period $2\pi/a$ and $B\cos(bt)$ has period $2\pi/b$. If there exist integers $m, n \in \mathbb{Z}$ such that $m2\pi/a = n2\pi/b = T$ (in other words, if $a/b = m/n \in \mathbb{Q}$) then for all t we have

$$\begin{aligned}
x(t+T) &= A\sin(a(t+T)) + B\cos(b(t+T)) \\
&= A\sin(a(t + m2\pi/a)) + B\cos(b(t + n2\pi/b)) \\
&= A\sin(at + m2\pi) + B\cos(bt + n2\pi) \\
&= A\sin(at) + B\cos(bt) \\
&= x(t),
\end{aligned}$$

and we conclude that $x(t)$ is periodic with period T.

Conversely, assume that $x(t)$ is periodic with some period T. In other words, assume that $x(t+T) = x(t)$ for all t. Then we also have $x''(t+T) = x''(t)$ for all t, and hence

$$x''(t+T) + b^2x(t+T) = x''(t) + b^2x(t), \text{ for all } t.$$

Since $x''(t) + b^2x(t) = (b^2 - a^2)A\sin(at)$, this implies that

$$(b^2 - a^2)A\sin(a(t+T)) = (b^2 - a^2)A\sin(at), \text{ for all } t.$$

I forgot to assume that $A \neq 0$ so I'll do that now. If $b^2 - a^2 = 0$ then certainly $a/b = \pm 1$, which is a rational number. Otherwise, we can cancel the constant $(b^2 - a^2)A$ from both sides to get

$$\sin(at + aT) = \sin(at), \text{ for all } t,$$

which implies that $aT = 2\pi m$ for some integer $m \in \mathbb{Z}$.

Similarly, we know that the function $x''(t) + a^2 x(t) = (a^2 - b^2)B\cos(bt)$ is periodic with period T, which implies that $\cos(bt + bT) = \cos(bt)$ for all t, and hence $bT = 2\pi n$ for some integer $n \in \mathbb{Z}$. We conclude that

$$\frac{2\pi m}{a} = T = \frac{2\pi n}{b}$$

and hence $a/b = m/n \in \mathbb{Q}$. ///

3. Damped Harmonic Oscillator. In class we found that the damped harmonic oscillator $x''(t) + x'(t) + x(t) = 0$ with initial condition $x'(0) = 0$ has solution

$$x(t) = \frac{2x(0)}{\sqrt{3}} \cdot e^{-t/2} \cdot \cos\left(\frac{\sqrt{3}}{2}t - \frac{\pi}{6}\right).$$

(a) For what values of t does $x(t) = \pm\frac{2x(0)}{\sqrt{3}}e^{-t/2}$?
(b) For what values of t does $x(t)$ have a local maximum/minimum?
(c) Graph the function $x(t)$ along with $\pm\frac{2x(0)}{\sqrt{3}}e^{-t/2}$.

For part (a), we are looking for t such that $\cos\left(\frac{\sqrt{3}}{2}t - \frac{\pi}{6}\right) = \pm 1$. We have $x(t) = \frac{2x(0)}{\sqrt{3}}e^{-t/2}$ when

$$\cos\left(\frac{\sqrt{3}}{2}t - \frac{\pi}{6}\right) = 1 \Longleftrightarrow \frac{\sqrt{3}}{2}t - \frac{\pi}{6} = 2\pi k \Longleftrightarrow t = \frac{\pi}{3\sqrt{3}} + \frac{4\pi}{\sqrt{3}}k$$

for some $k \in \mathbb{Z}$, and we have $x(t) = -\frac{2x(0)}{\sqrt{3}}e^{-t/2}$ when

$$\cos\left(\frac{\sqrt{3}}{2}t - \frac{\pi}{6}\right) = -1 \Longleftrightarrow \frac{\sqrt{3}}{2}t - \frac{\pi}{6} = \pi + 2\pi\ell \Longleftrightarrow t = \frac{7\pi}{3\sqrt{3}} + \frac{4\pi}{\sqrt{3}}\ell$$

for some $\ell \in \mathbb{Z}$.

For part (b) we first compute the derivative using the product rule:

$$\begin{aligned}
x'(t) &= \frac{2x(0)}{\sqrt{3}}\left[-\frac{1}{2}e^{-t/2} \cdot \cos\left(\frac{\sqrt{3}}{2}t - \frac{\pi}{6}\right) - \frac{\sqrt{3}}{2}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t - \frac{\pi}{6}\right)\right] \\
&= -\frac{2x(0)}{\sqrt{3}}e^{-t/2}\left[\frac{1}{2}\cos\left(\frac{\sqrt{3}}{2}t - \frac{\pi}{6}\right) + \frac{\sqrt{3}}{2}\sin\left(\frac{\sqrt{3}}{2}t - \frac{\pi}{6}\right)\right] \\
&= -\frac{2x(0)}{\sqrt{3}}e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t - \frac{\pi}{2}\right) \\
&= -\frac{2x(0)}{\sqrt{3}}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right).
\end{aligned}$$

[Wait, how did I do the second-last step? I used the angle-sum identity

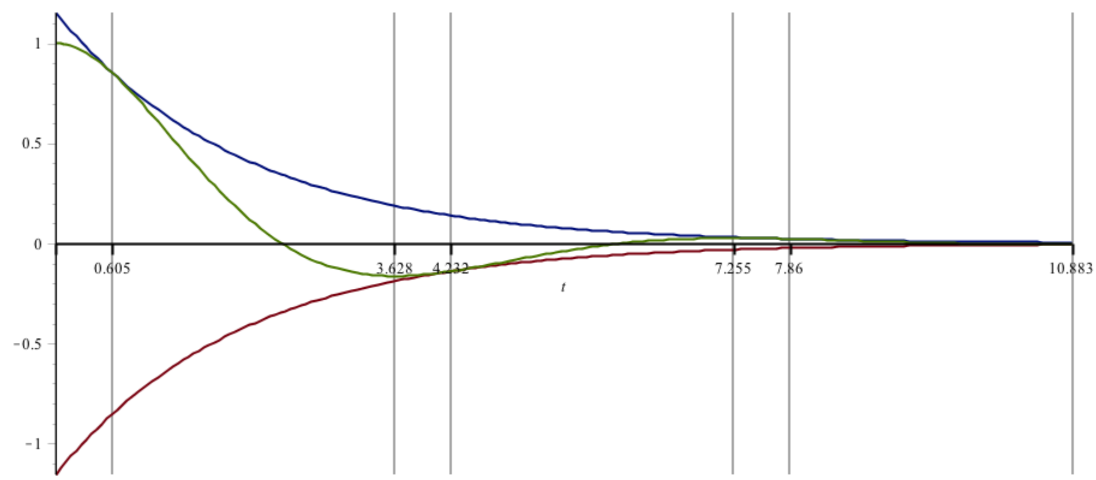
$$\cos\left(\frac{\sqrt{3}}{2}t - \frac{\pi}{6} + \varphi\right) = \cos\varphi \cdot \cos\left(\frac{\sqrt{3}}{2}t - \frac{\pi}{6}\right) - \sin\varphi \cdot \sin\left(\frac{\sqrt{3}}{2}t - \frac{\pi}{6}\right)$$

and then I solved the equations $\cos\varphi = \frac{1}{2}$ and $\sin\varphi = -\frac{\sqrt{3}}{2}$ to get $\varphi = -\frac{\pi}{3}$.] Now assume that $x(0) \neq 0$. Since $e^{-t/2}$ is never zero we conclude that

$$x'(t) = 0 \iff \sin\left(\frac{\sqrt{3}}{2}t\right) = 0 \iff \frac{\sqrt{3}}{2}t = \pi m \iff t = \frac{2\pi}{\sqrt{3}}m$$

for some $m \in \mathbb{Z}$. **Even** m correspond to local maxima and **odd** m correspond to local minima.

For part (c) let's choose $x(0) = 1$. Then here are the graphs of $x(t)$ and $\pm\frac{2}{\sqrt{3}}e^{-t/2}$ from $t = 0$ to $t = \frac{6\pi}{\sqrt{3}}$. The vertical lines show the locations of local maxima/minima and where the curves touch. Note that there is a slight delay after each local minimum/maximum until the curves touch. How much of a delay? Exactly $\frac{\pi}{3\sqrt{3}} \approx 0.6046$.



[Now you know everything you ever wanted to know about $x''(t) + x'(t) + x(t) = 0$.]

4. Hyperbolic Functions. Recall the definition of the hyperbolic functions:

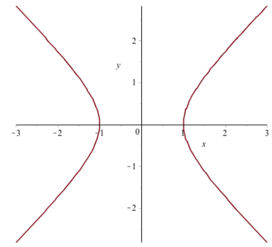
$$\cosh(t) := \frac{e^t + e^{-t}}{2} \qquad \text{and} \qquad \sinh(t) := \frac{e^t - e^{-t}}{2}.$$

(a) Verify that $\cosh^2(t) - \sinh^2(t) = 1$ for all $t \in \mathbb{R}$.
(b) Show that the parametrized curve $\mathbf{x}(t) = (\cosh(t), \sinh(t))$ is one branch of a hyperbola.
(c) Compute the velocity vector $\mathbf{x}'(t)$ at time t.
(d) Find a formula for the speed at time t. Is it constant?

For part (a) we have

$$\begin{aligned}\cosh^2(t) - \sinh^2(t) &= \frac{1}{4}(e^t + e^{-t})^2 - \frac{1}{4}(e^t - e^{-t})^2 \\ &= \frac{1}{4}(e^{2t} + 2 + e^{-2t}) - \frac{1}{4}(e^{2t} - 2 + e^{-2t}) \\ &= \frac{1}{4}(2) - \frac{1}{4}(-2) \\ &= 1.\end{aligned}$$

For part (b), recall that $x^2 - y^2 = 1$ is the equation of a hyperbola:



If we define $\mathbf{x}(t) = (x(t), y(t)) := (\cosh(t), \sinh(t))$ then from part (a) we know that $x(t)^2 - y(t)^2 = 1$, so the point $\mathbf{x}(t)$ is always on the hyperbola. Since $\mathbf{x}(0) = (1, 0)$ is on the right branch and since $\mathbf{x}(t)$ is a continuous function of t, we know that $\mathbf{x}(t)$ is always on the right branch. We also know that for $t_1 \neq t_2$ we have $\mathbf{x}(t_1) \neq \mathbf{x}(t_2)$ so the curve never backtracks. It must therefore trace out the whole right branch of the parabola. (We also need to verify that the velocity never goes to zero, which we'll do below.)

For part (c) we first compute

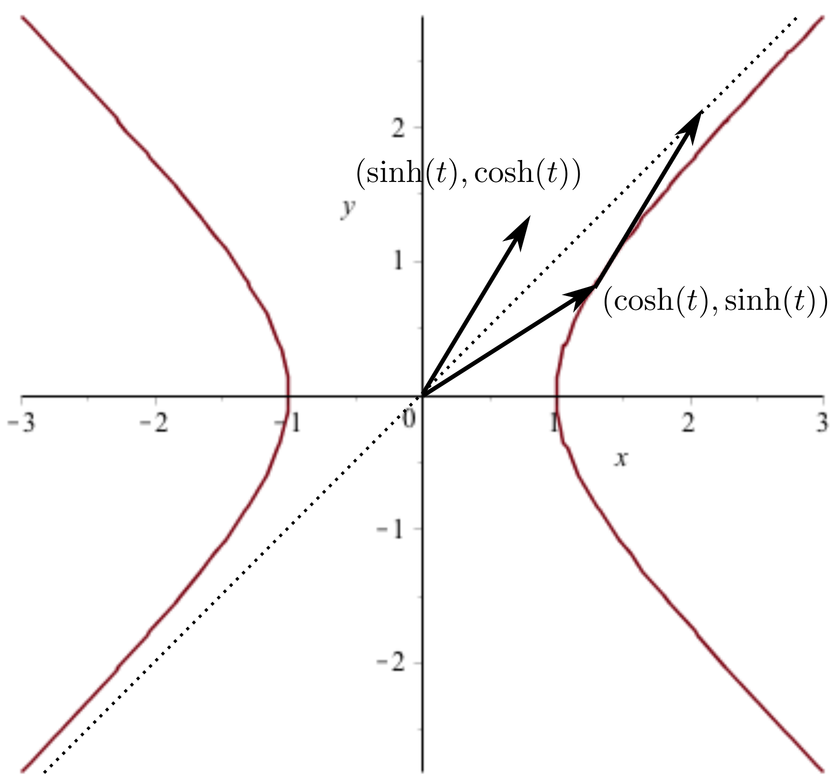
$$\frac{d}{dt}\cosh(t) = \frac{d}{dt}\frac{e^t + e^{-t}}{2} = \frac{e^t - e^{-t}}{2} = \sinh(t)$$

and

$$\frac{d}{dt}\sinh(t) = \frac{d}{dt}\frac{e^t - e^{-t}}{2} = \frac{e^t + e^{-t}}{2} = \cosh(t)$$

Then the velocity of the curve $\mathbf{x}(t) = (\cosh(t), \sinh(t))$ at time t is

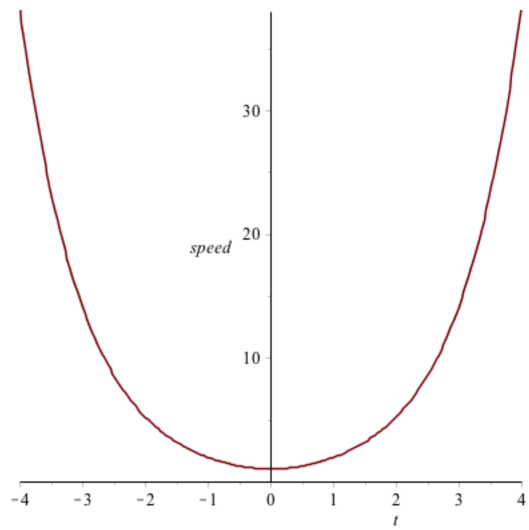
$$\mathbf{x}'(t) = (x'(t), y'(t)) = (\sinh(t), \cosh(t)).$$



The (squared) speed of the curve at time t is given by

$$\begin{aligned}
|\mathbf{x}'(t)|^2 &= x'(t)^2 + y'(t)^2 \\
&= \cosh^2(t) + \sinh^2(t) \\
&= \frac{1}{4}(e^t + e^{-t})^2 + \frac{1}{4}(e^t - e^{-t})^2 \\
&= \frac{1}{4}(e^{2t} + 2 + e^{-2t}) + \frac{1}{4}(e^{2t} - 2 + e^{-2t}) \\
&= \frac{1}{2}(e^{2t} + e^{-2t}) \\
&= \cosh(2t).
\end{aligned}$$

This is certainly **not** constant. It goes to $+\infty$ as $t \to +\infty$ or $t \to -\infty$ and it has a global minimum of $|\mathbf{x}'(t)| = 1$ at $t = 0$. Here is a graph of the speed $|\mathbf{x}'(t)|$ for $-4 < t < 4$.



5. Eigenvalues. Consider a matrix $A = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$ and a vector $\mathbf{x} = (x, y)$. We say that $\mathbf{x} \neq (0,0)$ is an **eigenvector** of A if there exists a constant λ such that

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}.$$

Explicitly try to solve this system of two linear equations to obtain x, y in terms of a, b, c, d, λ, and show that it has a solution only if $(a - \lambda)(d - \lambda) - bc = 0$. [Hint: Assume that it **has** a solution $(x, y) \neq (0, 0)$ and try to prove that $(a - \lambda)(d - \lambda) - bc = 0$. If $c \neq 0$ then subtract $(d-\lambda)/c$ times the first equation from the second equation to obtain $bx-(a-\lambda)(d-\lambda)x/c = 0$. Since $c \neq 0$ we must have $x \neq 0$ (why?), which implies that $(a - \lambda)(d - \lambda) - bc = 0$. If $c = 0$ but $b \neq 0$ then subtract $(a-\lambda)/b$ times the second equation from the first equation. Conclude again that $(a - \lambda)(d - \lambda) - bc = 0$. Finally, if both $c = 0$ and $b = 0$, show that we must **still** have $(a - \lambda)(d - \lambda) - bc = 0$. There is no escape!]

Proof. Assume that the system has a solution (x, y) with x and y **not both zero**, and write the matrix equation as a system of two linear equations

$$(a-\lambda)x + cy = 0 \tag{1}$$

$$bx + (d-\lambda)y = 0. \tag{2}$$

If $c \neq 0$ then we can subtract $(d-\lambda)/c$ times equation (1) from equation (2) to obtain

$$\begin{aligned} (bx + (d-\lambda)y) - \frac{(d-\lambda)}{c}((a-\lambda)x + cy) &= 0 - 0 \\ bx + (d-\lambda)y - \frac{(d-\lambda)(a-\lambda)}{c}x - (d-\lambda)y &= 0 \\ bx - \frac{(d-\lambda)(a-\lambda)}{c}x &= 0 \\ bcx - (d-\lambda)(a-\lambda)x &= 0 \\ (bc - (d-\lambda)(a-\lambda))x &= 0. \end{aligned}$$

Note that x cannot be zero, otherwise equation (1) says $cy = 0$ which since $c \neq 0$ implies $y = 0$. But then x and y are both zero; contradiction. Thus we can cancel x in the previous equation to conclude that

$$bc - (a-\lambda)(d-\lambda) = 0,$$

as desired.

If $c = 0$ but $b \neq 0$ then we can subtract $(a-\lambda)/b$ times equation (2) from equation (1) to obtain

$$\begin{aligned} ((a-\lambda)x + cy) - \frac{(a-\lambda)}{b}(bx + (d-\lambda)y) &= 0 - 0 \\ (a-\lambda)x + cy - (a-\lambda)x - \frac{(a-\lambda)(d-\lambda)}{b}y &= 0 \\ cy - \frac{(a-\lambda)(d-\lambda)}{b}y &= 0 \\ bcy - (a-\lambda)(d-\lambda)y &= 0 \\ (bc - (a-\lambda)(d-\lambda))y &= 0. \end{aligned}$$

Note that y cannot be be zero, otherwise equation (2) says $bx = 0$ shich since $b \neq 0$ implies $x = 0$. But then x and y are both zero; contradiction. Thus we can cancel y in the above equation to conclude that

$$bc - (a-\lambda)(d-\lambda) = 0,$$

as desired.

Finally, if $c = 0$ and $b = 0$ then equations (1) and (2) become

$$\begin{aligned} (a-\lambda)x &= 0 \\ (d-\lambda)y &= 0. \end{aligned}$$

Since x and y are not both zero, this implies that at least one of $(a-\lambda)$ and $(d-\lambda)$ must be zero, and hence $(a-\lambda)(d-\lambda) = 0$. This implies that

$$(a-\lambda)(d-\lambda) - bc = 0 - 0 = 0,$$

as desired. □

[If you've taken linear algebra then you probably already knew this. But maybe you never proved it before. Now you have. From now on you can use it with impunity.]