Math 592 **Spring 2015**
Homework 2 **Drew Armstrong**

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1. Beats Again. Show that the phenomenon of beats is independent of "phase shifts". [Hint: Consider the superposition $\sin(f_1 \cdot 2\pi t + \varphi) + \sin(f_2 \cdot 2\pi t + \mu)$.]

2. Galileo's Theory of Dissonance. Let $a, b, A, B \in \mathbb{R}$. Show that the function

$$x(t) = A\sin(at) + B\cos(bt)$$

is periodic if and only if the number a/b equals a fraction of whole numbers. In this case, what is the period? Galileo believed that this is the reason we prefer ratios of small whole numbers: so that our eardrum is not "kept in perpetual torment".

3. Damped Harmonic Oscillator. In class we found that the damped harmonic oscillator $x''(t) + x'(t) + x(t) = 0$ with initial condition $x'(0) = 0$ has solution

$$x(t) = \frac{2x(0)}{\sqrt{3}} \cdot e^{-t/2} \cdot \cos\left(\frac{\sqrt{3}}{2}t - \frac{\pi}{6}\right).$$

(a) For what values of t does $x(t) = \pm\frac{2x(0)}{\sqrt{3}}e^{-t/2}$?
(b) For what values of t does $x(t)$ have a local maximum/minimum?
(c) Graph the function $x(t)$ along with $\pm\frac{2x(0)}{\sqrt{3}}e^{-t/2}$.

4. Hyperbolic Functions. Recall the definition of the hyperbolic functions:

$$\cosh(t) := \frac{e^t + e^{-t}}{2} \qquad \text{and} \qquad \sinh(t) := \frac{e^t - e^{-t}}{2}.$$

(a) Verify that $\cosh^2(t) - \sinh^2(t) = 1$ for all $t \in \mathbb{R}$.
(b) Show that the parametrized curve $\mathbf{x}(t) = (\cosh(t), \sinh(t))$ is one branch of a hyperbola.
(c) Compute the velocity vector $\mathbf{x}'(t)$ at time t.
(d) Find a formula for the speed at time t. Is it constant?

5. Eigenvalues. Consider a matrix $A = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$ and a vector $\mathbf{x} = (x, y)$. We say that $\mathbf{x} \neq (0,0)$ is an **eigenvector** of A if there exists a constant λ such that

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}.$$

Explicitly try to solve this system of two linear equations to obtain x, y in terms of a, b, c, d, λ, and show that it has a solution only if $(a-\lambda)(d-\lambda) - bc = 0$. [Hint: Assume that it **has** a solution $(x, y) \neq (0,0)$ and try to prove that $(a-\lambda)(d-\lambda) - bc = 0$. If $c \neq 0$ then subtract $(d-\lambda)/c$ times the first equation from the second equation to obtain $bx - (a-\lambda)(d-\lambda)x/c = 0$. Since $c \neq 0$ we must have $x \neq 0$ (why?), which implies that $(a-\lambda)(d-\lambda) - bc = 0$. If $c = 0$ but $b \neq 0$ then subtract $(a-\lambda)/b$ times the second equation from the first equation. Conclude again that $(a-\lambda)(d-\lambda) - bc = 0$. Finally, if both $c = 0$ and $b = 0$, show that we must **still** have $(a-\lambda)(d-\lambda) - bc = 0$. There is no escape!]