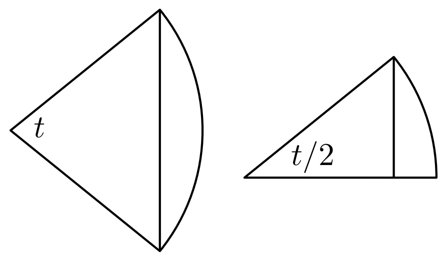
Math 592 **Spring 2015**
Homework 1 **Drew Armstrong**

0. Compute the length of a chord of the unit circle subtended by an arc of length t.

Consider the chord and the half-chord subtended by an arc of length t:



If $\text{crd}(t)$ is the length of the chord then $\frac{1}{2}\text{crd}(t)$ is the length of the half-chord. Since the half-chord is the opposite side of a right triangle with angle $t/2$ and hypotenuse of length 1 (the circle has radius 1) we conclude that $\frac{1}{2}\text{crd}(t) = \sin(t/2)$, and hence

$$\text{crd}(t) = 2\sin(t/2).$$

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1. Given an arbitrary matrix $A = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$ we can define a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ by $\mathbf{x} \mapsto A\mathbf{x}$, in other words,

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + cy \\ bx + dy \end{pmatrix}.$$

Prove that this is a linear function.

For all vectors $(x, y), (x', y') \in \mathbb{R}^2$ and all constants $\alpha, \beta \in \mathbb{R}$, we have

$$\begin{aligned}
\alpha \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \beta \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} \alpha(ax + cy) \\ \alpha(bx + dy) \end{pmatrix} + \begin{pmatrix} \beta(ax' + cy') \\ \beta(bx' + dy') \end{pmatrix} \\
&= \begin{pmatrix} \alpha(ax + cy) + \beta(ax' + cy') \\ \alpha(bx + dy) + \beta(bx' + dy') \end{pmatrix} \\
&= \begin{pmatrix} a(\alpha x + \beta x') + c(\alpha y + \beta y') \\ b(\alpha x + \beta x') + d(\alpha y + \beta y') \end{pmatrix} \\
&= \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} \alpha x + \beta x' \\ \alpha y + \beta y' \end{pmatrix} \\
&= \begin{pmatrix} a & c \\ b & d \end{pmatrix} \left[\alpha \begin{pmatrix} x \\ y \end{pmatrix} + \beta \begin{pmatrix} x' \\ y' \end{pmatrix} \right],
\end{aligned}$$

as desired. ///

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear function and consider the standard basis of $\mathbb{R}^2$ consisting of $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$. If $f(\mathbf{e}_1) = (a, b)$ and $f(\mathbf{e}_2) = (c, d)$ then we define the matrix

$$[f] = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Given $\mathbf{x} \in \mathbb{R}^2$ we will write $[\mathbf{x}]$ for the corresponding column vector. Then we define the product of a matrix and a column by $[f][\mathbf{x}] = [f(\mathbf{x})]$. [Why do we do this?]

2. Let f and g be linear functions from $\mathbb{R}^2$ to $\mathbb{R}^2$.

(a) Prove that the composite $f \circ g : \mathbb{R}^2 \to \mathbb{R}^2$ is also linear.

(b) We define the matrix product by $[f][g] := [f \circ g]$. If $[f] = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$ and $[g] = \left(\begin{smallmatrix} a' & c' \\ b' & d' \end{smallmatrix}\right)$, use the definition to compute the matrix product $[f][g]$.

(a) Assume that f and g are linear functions $\mathbb{R}^2 \to \mathbb{R}^2$. Then for all vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ and all constants $\alpha, \beta \in \mathbb{R}$ we have

$$\begin{aligned}(f \circ g)(\alpha\mathbf{x} + \beta\mathbf{y}) &= f(g(\alpha\mathbf{x} + \beta\mathbf{y})) \\ &= f(\alpha g(\mathbf{x}) + \beta g(\mathbf{y})) \\ &= \alpha f(g(\mathbf{x})) + \beta f(g(\mathbf{y})) \\ &= \alpha(f \circ g)(\mathbf{x}) + \beta(f \circ g)(\mathbf{y}),\end{aligned}$$

hence $f \circ g$ is a linear function. ///

(b) Since $f \circ g$ is linear it can be represented by a matrix, which we call $[f][g]$. We will compute this matrix, assuming that $[f] = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$ and $[g] = \left(\begin{smallmatrix} a' & c' \\ b' & d' \end{smallmatrix}\right)$. To do this, we consider any vector $(x, y) \in \mathbb{R}^2$. Then applying $f \circ g$ to (x, y) (writing everything in standard coordinates) gives

$$\begin{aligned}(f \circ g)\begin{pmatrix} x \\ y \end{pmatrix} &= f\left(g\begin{pmatrix} x \\ y \end{pmatrix}\right) \\ &= f\left(\begin{pmatrix} a' & c' \\ b' & d' \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right) \\ &= f\begin{pmatrix} a'x + c'y \\ b'x + d'y \end{pmatrix} \\ &= \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} a'x + c'y \\ b'x + d'y \end{pmatrix} \\ &= \begin{pmatrix} a(a'x + c'y) + c(b'x + d'y) \\ b(a'x + c'y) + d(b'x + d'y) \end{pmatrix} \\ &= \begin{pmatrix} (aa' + cb')x + (ac' + cd')y \\ (ba' + db')x + (bc' + dd')y \end{pmatrix} \\ &= \begin{pmatrix} aa' + cb' & ac' + cd' \\ ba' + db' & bc' + dd' \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.\end{aligned}$$

We conclude that $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)\left(\begin{smallmatrix} a' & c' \\ b' & d' \end{smallmatrix}\right) = [f][g] = [f \circ g] = \left(\begin{smallmatrix} aa'+cb' & ac'+cd' \\ ba'+db' & bc'+dd' \end{smallmatrix}\right)$. ///

[Remark: That is a computation that everyone should do at least once in their life. Now you have.]

3. Let $R_t : \mathbb{R}^2 \to \mathbb{R}^2$ be the (linear) function that rotates the plane counterclockwise by angle t. Recall that we can express this in coordinates by

$$[R_t] = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

(a) Explain why $[R_t]^3 = [R_{3t}]$ without doing any work.

(b) Use part (a) to express $\cos(3t)$ as a polynomial in $\cos(t)$. This is an example of a Chebyshev polynomial of the first kind.

(a) The function R_t rotates the plane counterclockwise by angle t. Therefore the function $R_t \circ R_t \circ R_t$ rotates the plane counterclockwise by angle $3t$, i.e., we have $R_t \circ R_t \circ R_t = R_{3t}$. Writing this in coordinates gives

$$[R_t]^3 = [R_t][R_t][R_t] = [R_t \circ R_t \circ R_t] = [R_{3t}].$$

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(b) The equation from part (a) says that

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}^3 = \begin{pmatrix} \cos(3t) & -\sin(3t) \\ \sin(3t) & \cos(3t) \end{pmatrix}.$$

On the other hand, using the formula from Problem 2(b) gives

$$\begin{aligned} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}^3 &= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \\ &= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\begin{pmatrix} \cos^2 t - \sin^2 t & -2\sin t\cos t \\ 2\sin t\cos t & \cos^2 t - \sin^2 t \end{pmatrix} \\ &= \begin{pmatrix} \cos t(\cos^2 t - \sin^2 t) - 2\sin^2 t\cos t & -2\sin t\cos^2 t - \sin t(\cos^2 t - \sin^2 t) \\ 2\sin t\cos^2 t + \sin t(\cos^2 t - \sin^2 t) & \cos t(\cos^2 t - \sin^2 t) - 2\sin^2 t\cos t \end{pmatrix} \end{aligned}$$

Comparing the top-left entries of the two matrices gives

$$\begin{aligned} \cos(3t) &= \cos t(\cos^2 t - \sin^2 t) - 2\sin^2 t\cos t \\ &= \cos^3 t - \sin^2 t\cos t - 2\sin^2 t\cos t \\ &= \cos^3 t - 3\sin^2 t\cos t \\ &= \cos^3 t - 3(1-\cos^2 t)\cos t \\ &= \cos^3 t + 3\cos^3 t - 3\cos t \\ &= 4\cos^3 t - 3\cos t. \end{aligned}$$

This polynomial is called a Chebyshev polynomial of the first kind. We use the notation $T_3(x) = 4x^3 - 3x$. The general polynomial $T_n(x)$ expresses $\cos(nt)$ as a polynomial in $\cos t$; that is, we have $T_n(\cos t) = \cos(nt)$. These polynomials appear everywhere and have lots of uses. ///

4. Use the "angle sum fomulas" to verify the following trigonometric identities.

(a) $2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$
(b) $2\cos\alpha\cos\beta = \cos(\alpha-\beta) + \cos(\alpha+\beta)$

Recall that the angle sum formulas say

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{1}$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{2}$$

From these, using the fact that $\cos(-\beta) = \cos\beta$ and $\sin(-\beta) = -\sin\beta$, we obtain

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \tag{3}$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \tag{4}$$

For part (a) we subtract formula (1) from formula (3) to get

$$\begin{aligned} \cos(\alpha-\beta) - \cos(\alpha+\beta) &= (\cos\alpha\cos\beta + \sin\alpha\sin\beta) - (\cos\alpha\cos\beta - \sin\alpha\sin\beta) \\ &= 2\sin\alpha\sin\beta. \end{aligned}$$

For part (b) we add formula (1) to formula (3) to get

$$\begin{aligned}\cos(\alpha-\beta)+\cos(\alpha+\beta) &= (\cos\alpha\cos\beta+\sin\alpha\sin\beta)+(\cos\alpha\cos\beta-\sin\alpha\sin\beta)\\ &= 2\cos\alpha\cos\beta.\end{aligned}$$

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5. Use the identities from Problem 4 to verify the following integrals.

(a) $\displaystyle\int_0^{2\pi}\sin(mt)\sin(nt)\,dt = \begin{cases}\pi & m=n\neq 0\\ 0 & \text{otherwise}\end{cases}$

(b) $\displaystyle\int_0^{2\pi}\cos(mt)\cos(nt)\,dt = \begin{cases}2\pi & m=n=0\\ \pi & m=n\neq 0\\ 0 & \text{otherwise}\end{cases}$

(a) First we use Problem 4(a) to write

$$\sin(mt)\sin(nt) = \frac{1}{2}[\cos((m-n)t)-\cos((m+n)t)].$$

If $m-n\neq 0$ and $m+n\neq 0$ then we integrate to obtain

$$\begin{aligned}\int_0^{2\pi}\sin(mt)\sin(nt)\,dt &= \frac{1}{2}\int_0^{2\pi}[\cos((m-n)t)-\cos((m+n)t)]\,dt\\ &= \frac{1}{2}\left[\frac{\sin((m-n)t)}{m-n}+\frac{\sin((m+n)t)}{m+n}\right]_0^{2\pi}\\ &= \frac{1}{2}\left[\frac{\sin((m-n)2\pi)}{m-n}+\frac{\sin((m+n)2\pi)}{m+n}\right].\end{aligned}$$

If $m-n$ and $m+n$ are both half-integers then $\sin((m-n)2\pi)=\sin((m+n)2\pi)=0$, so the integral evaluates to zero. We are probably assuming that m and n are **non-negative integers**, but it's nice to pay attention to how the assumption is used.

If $m=n=0$ then the integral is $\int_0^{2\pi}\sin(0)\sin(0)\,dt=\int_0^{2\pi}0\,dt=0$.

If $m=n\neq 0$ then the integral is

$$\begin{aligned}\int_0^{2\pi}\sin(mt)\sin(mt)\,dt &= \frac{1}{2}\int_0^{2\pi}[\cos(0)-\cos(2mt)]\,dt\\ &= \frac{1}{2}\left[t+\frac{\sin(2mt)}{2}\right]_0^{2\pi}\\ &= \frac{1}{2}\left[2\pi+\frac{\sin(m4\pi t)}{2}\right]\\ &= \pi.\end{aligned}$$

In the last step I assumed that m is a quarter-integer so that $\sin(m4\pi)=0$.

Finally, if $m = -n \neq 0$ then we have

$$\begin{aligned}
\int_0^{2\pi} \sin(mt)\sin(-mt)\,dt &= \frac{1}{2}\int_0^{2\pi} [\cos(2mt) - \cos(0)]\,dt \\
&= \frac{1}{2}\left[\frac{\sin(2mt)}{2} - t\right]_0^{2\pi} \\
&= \frac{1}{2}\left[\frac{\sin(m4\pi t)}{2} - 2\pi\right] \\
&= -\pi.
\end{aligned}$$

Again, I assumed that m is a quarter-integer. This case does not appear in the problem, because, again, in the problem I was probably assuming that m and n are **non-negative integers**. ///

(b) Here I'll just assume at the outset that m and n are non-negative integers, and ignore the other cases. We will use the following identity from Problem 4(b):

$$\cos(mt)\cos(nt) = \frac{1}{2}[\cos((m-n)t) + \cos((m+n)t)].$$

If $m \neq n$ then we have

$$\begin{aligned}
\int_0^{2\pi} \cos(mt)\cos(nt)\,dt &= \frac{1}{2}\int_0^{2\pi} [\cos((m-n)t) + \cos((m+n)t)]\,dt \\
&= \frac{1}{2}\left[\frac{\sin((m-n)t)}{m-n} + \frac{\sin((m+n)t)}{m+n}\right]_0^{2\pi} \\
&= 0.
\end{aligned}$$

If $m = n = 0$ then we have

$$\int_0^{2\pi} \cos(0)\cos(0)\,dt = \int_0^{2\pi} 1\,dt = 2\pi.$$

If $m = n \neq 0$ then we have

$$\begin{aligned}
\int_0^{2\pi} \cos(mt)\cos(mt)\,dt &= \frac{1}{2}\int_0^{2\pi} [\cos(0) + \cos(2mt)]\,dt \\
&= \frac{1}{2}\left[t + \frac{\sin(2mt)}{2}\right]_0^{2\pi} \\
&= \frac{1}{2}[2\pi + 0] \\
&= \pi.
\end{aligned}$$

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[Remark: It is not clear right now what these formulas are good for. We will see later that they are the foundation of the theory of Fourier series.]