Math 562/662 **Spring 2019**
Homework 2 **Drew Armstrong**

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1. Addition vs. Multiplication. Prove that following properties hold in any ring.

(a) $0a = 0$,
(b) $a(-b) = (-a)b = -(ab)$,
(c) $(-a)(-b) = ab$.

2. Characteristic of a Ring. Let R be a ring and let $R' \subseteq R$ be the smallest subring. Recall that there exists a unique ring homomorphism $\iota_R : \mathbb{Z} \to R$ from the integers.

(a) Prove that $R' \cong \mathbb{Z}/n\mathbb{Z}$ for some integer $n \geq 0$, which we call the *characteristic* of R:
$$\text{char}(R) = n.$$
[Hint: Apply the First Isomorphism Theorem to ι_R.]
(b) If $\varphi : R \to S$ is any ring homomorphism prove that $\text{char}(S)$ divides $\text{char}(R)$. [Hint: By uniqueness we know that $\iota_S = \varphi \circ \iota_R$. Consider the kernel.]
(c) Next let R be an *integral domain*, which means that R has no *zero divisors*:
$$\forall a, b \in R, (ab = 0) \Rightarrow (a = 0 \text{ or } b = 0).$$
In this case prove that $\text{char}(R) = 0$ or $\text{char}(R) = p$ for some prime p.
(d) Finally, let $\mathbb{F}$ ba a field and let $\mathbb{F}' \subseteq \mathbb{F}$ be the smallest subfield. Prove that
$$\mathbb{F}' \cong \mathbb{Q} \qquad \text{or} \qquad \mathbb{F}' \cong \mathbb{Z}/p\mathbb{Z} \text{ for some prime } p.$$

3. Chinese Remainder Theorem, Part II. Let R be a ring. For any ideals $I, J \subseteq R$ we define the *product ideal*:
$$IJ := \text{intersection of all ideals that contain } \{ab : a \in I, b \in J\}.$$

(a) Prove that $IJ \subseteq I \cap J$.
(b) We say that $I, J \subseteq R$ are *coprime* if $I + J = R$. In this case show that $I \cap J \subseteq IJ$, and hence $IJ = I \cap J$. [Hint: Since $1 \in I + J$ we have $1 = x + y$ for some $x \in I$ and $y \in J$.]
(c) If $I, J \subseteq R$ are coprime, prove that the obvious map $(a + IJ) \mapsto (a + I, a + J)$ defines an isomorphism of rings:
$$\frac{R}{IJ} \cong \frac{R}{I} \times \frac{R}{J}.$$
[Hint: The hardest part is surjectivity. Use the same trick that you used when $R = \mathbb{Z}$.]

4. Ring Isomorphism Theorems. Let R be a ring and let $I \subseteq R$ be an ideal.

(a) For any additive subgroup $I \subseteq S \subseteq R$ prove that
$$S \subseteq R \text{ is a subring} \quad \Longleftrightarrow \quad S/I \subseteq R/I \text{ is a subring}.$$
(b) For any subring $S \subseteq R$ prove that we have an isomorphism of rings:
$$\frac{S}{S \cap I} \cong \frac{S + I}{I}.$$
[Hint: Consider the ring homomorphism $\varphi : S \to R/I$ defined by $\varphi(a) = a + I$.]

(c) For any additive subgroup $I \subseteq J \subseteq R$ prove that

$$J \subseteq R \text{ is an ideal} \quad \Longleftrightarrow \quad J/I \subseteq R/I \text{ is an ideal},$$

in which case we have an isomorphism of rings:

$$\frac{R/I}{J/I} \cong \frac{R}{J}.$$

[Hint: Consider the ring homomorphism $\varphi : R/I \to R/J$ defined by $\varphi(a + I) = a + J$.]

5. Descartes' Factor Theorem. Let $E \supseteq R$ be any ring extension and let $f(x) \in R[x]$ be any polynomial with coefficients in R.

(a) For any element $\alpha \in E$ prove that $f(\alpha) = 0$ if and only if there exists a polynomial $h(x) \in E[x]$ with coefficients in E such that $f(x) = (x-\alpha)h(x)$ and $\deg(h) = \deg(f)-1$. [Hint: For all integers $n \geq 2$ observe that

$$x^n - \alpha^n = (x - \alpha)(x^{n-1} + x^{n-2}\alpha + \cdots + x\alpha^{n-2} + \alpha^{n-1}) \in E[x].$$

Now consider the polynomial $f(x) - f(\alpha) \in E[x]$.]

(b) **Counting Roots.** If E is an *integral domain*, use the result of part (a) to prove that any polynomial $f(x) \in R[x]$ has at most $\deg(f)$ distinct roots in E.

(c) **A Non-Example.** Let $E = R = \mathbb{Z}/8\mathbb{Z}$ and consider the polynomial $x^2 - 1$. How many roots does this polynomial have? Why does this not contradict part (b)?

6. Prime and Maximal Ideals. Let R be a ring and let $I \subseteq R$ be an ideal.

(a) We say that I is a *maximal ideal* if

$$\text{for any ideal } J \subseteq R \text{ we have } (I \subsetneq J) \Rightarrow (J = R).$$

Prove that R/I is a field if and only if I is maximal.

(b) We say that I is a *prime ideal*

$$\text{for any } a, b \in R \text{ we have } (ab \in I) \Rightarrow (a \in I \text{ or } b \in I).$$

Prove that R/I is an integral domain if and only if I is prime.

(c) Prove that every maximal ideal is prime.

(d) Let $\mathbb{Z}[x]$ be the ring of polynomials over $\mathbb{Z}$ and consider the principal ideal

$$\langle x \rangle = \{xf(x) : f(x) \in \mathbb{Z}[x]\}.$$

Prove that $\langle x \rangle$ is prime but not maximal. [Hint: $\mathbb{Z}[x]/\langle x \rangle \cong \mathbb{Z}$.]