5/29/14

Review of 561/562

Algebra Prelim Tomorrow 3-6pm (here)

Last Time:
- Conjugation
- The Class Equation & Applications

Today: Matrices

Let X be a set with structure and let

$$\text{Aut}(X) = \{ \text{invertible structure preserving maps } X \to X \}.$$

Examples:

$$\text{Aut}(\text{set } X) = \text{Perm}(X)$$

$$\text{Aut}(\text{vector space } K^n) = GL(K^n)$$

If we choose a basis for K^n then we can identify $GL(K^n)$ with

$$GL_n(K) = \{ n \times n \text{ matrices with matrix multiplication} \}$$

Aut(inner product space $\mathbb{R}^n$) $= O(n)$.

Choose basis $e_1, \dots, e_n \in \mathbb{R}^n$ and use the inner product

$$x^t y = \sum_i x_i y_i .$$

Then $O(n) \leq GL_n(\mathbb{R})$ is defined by

$$O(n) = \{ A \in GL_n(\mathbb{R}) : (Ax)^t (Ay) = x^t y \;\; \forall x, y \in \mathbb{R}^n \}$$

i.e. $x^t y = (Ax)^t (Ay) = x^t A^t A y = x^t (A^t A) y$.

Putting $x = e_i$ & $y = e_j$ gives

$$\begin{aligned} i,j \text{ entry of } A^t A &= e_i^t (A^t A) e_j \\ &= e_i^t e_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \end{aligned}$$

$$\Longrightarrow \quad A^t A = I$$

Conclusion:

$$O(n) = \{ A \in GL_n(\mathbb{R}) : A^t A = I \}$$

Note : $A^tA = I$

$$\Rightarrow \det(A)^2 = \det(A)\det(A)$$
$$= \det(A^t)\det(A)$$
$$= \det(A^tA) = \det(I) = 1$$

$$\Rightarrow \det(A) = \pm 1.$$

Notation :

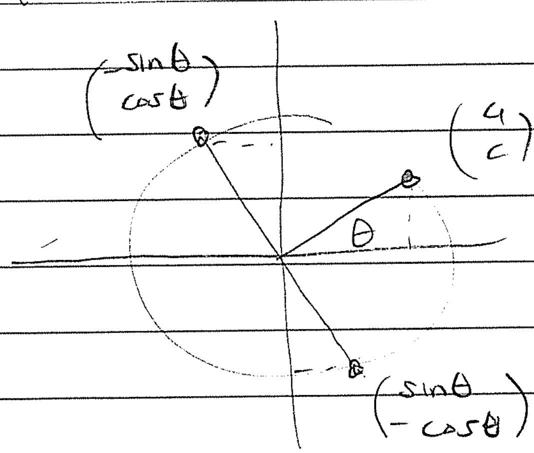
any elt. of det $= -1$.

$$O(n) = \underbrace{SO(n)}_{\det = +1} \cup \underbrace{R \cdot SO(n)}_{\det = -1}$$

Case $n = 2$:

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $A^tA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Since $\begin{pmatrix} a \\ c \end{pmatrix}\begin{pmatrix} a & c \end{pmatrix} = a^2 + c^2 = 1$ we have



$\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

for some θ.

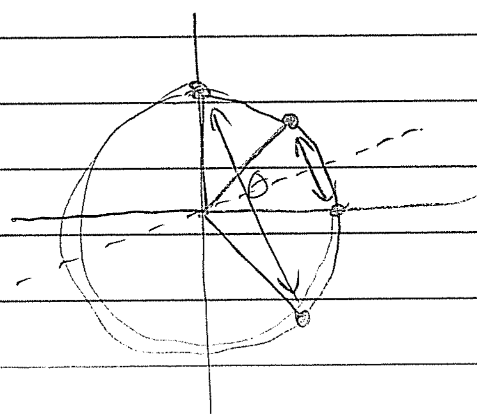
Then because $\begin{pmatrix} a \\ c \end{pmatrix} \begin{pmatrix} b & d \end{pmatrix} = 0$

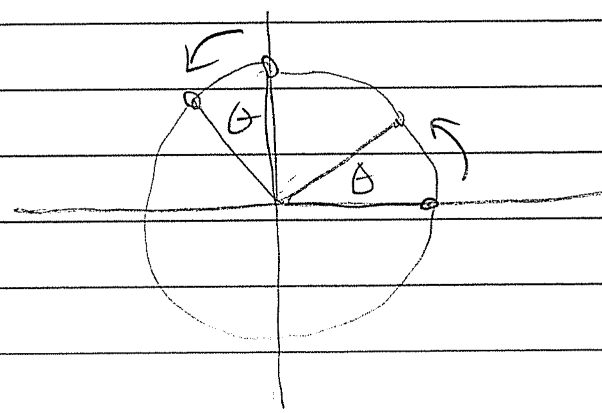
and $\begin{pmatrix} b \\ d \end{pmatrix} \begin{pmatrix} b & d \end{pmatrix} = 1$

there are <u>exactly</u> <u>two</u> choices:

det +1 $A = \rho_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ rotate θ c.c.w.

det -1 $A = R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ reflect across line of angle $\theta/2$





Exercises:

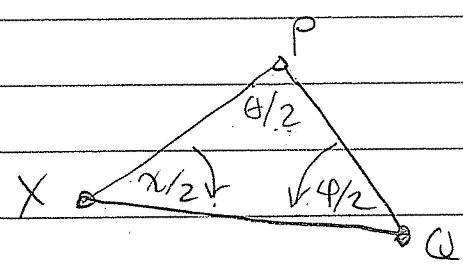
- $\rho_\alpha \rho_\beta = \rho_{\alpha+\beta}$ (de Moivre's Theorem)
- $R_\alpha R_\beta = \rho_{\alpha-\beta}$ (product of two reflections is rotation by <u>twice</u> the angle between them)

Now let $\mathrm{Isom}(\mathbb{R}^2) = \mathrm{Aut}($affine inner product space $\mathbb{R}^2)$ and let

ρ_θ^P = rotate c.c.w. by θ around point $P \in \mathbb{R}^2$.

Q: $\rho_\varphi^Q \circ \rho_\theta^P = ?$

A: Consider the triangle in $\mathbb{R}^2$



Let R_{PQ}, R_{PX}, R_{QX} be reflections in the sides of the triangle. Then

$$\rho_\varphi^Q \circ \rho_\theta^P = (R_{QX} \circ R_{PQ}) \circ (R_{PQ} \circ R_{PX})$$

$$= R_{QX} \circ \cancel{(R_{PQ} \circ R_{PQ})} \circ R_{PX}$$

$$= R_{QX} \circ R_{PX}$$

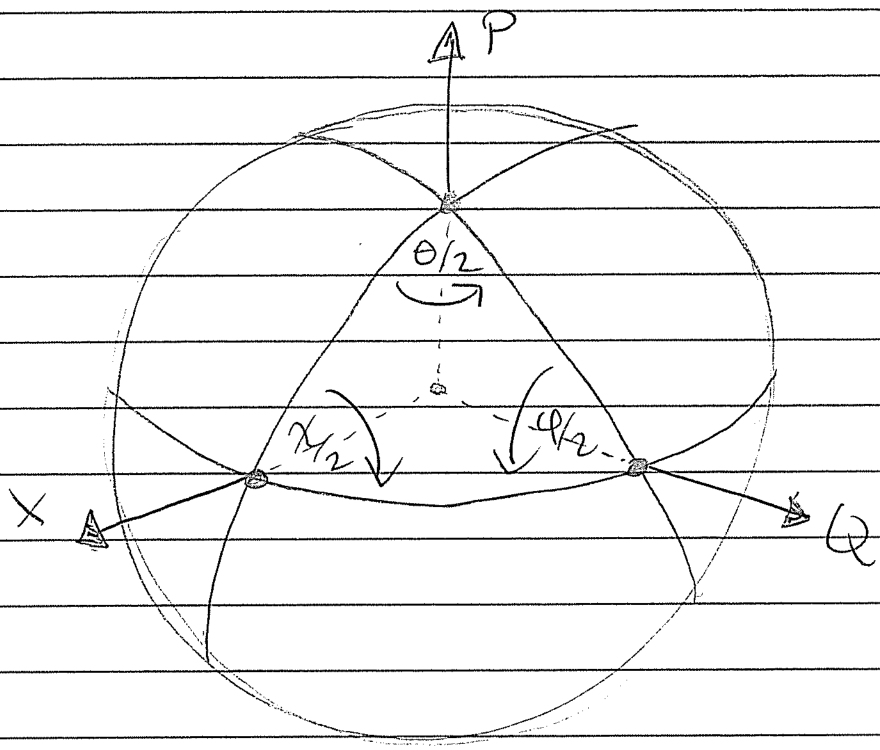
$$= (R_{PX} \circ R_{QX})^{-1}$$

$$= (\rho_x^X)^{-1} = \rho_{-x}^X$$

rotation clock<u>wise</u> by x around X

Summary: rotation $\circ$ rotation = rotation.

The same theorem is true in the group $O(3)$ using the same proof



$\blacksquare$

Finally, think about

$\mathrm{Isom}(\mathbb{R}^n) = \mathrm{Aut}($affine inner product space $\mathbb{R}^n)$

$= \mathrm{Aut}($"Euclidean space" $\mathbb{R}^n)$.

It contains a subgroup of "translations"

$$t_\alpha : \mathbb{R}^n \to \mathbb{R}^n \quad \text{for } \alpha \in \mathbb{R}^n.$$
$$x \mapsto x + \alpha$$

Let $T = \{t_\alpha : \alpha \in \mathbb{R}^n\} \leq \mathrm{Isom}(\mathbb{R}^n)$ and note that

$$T \cong (\mathbb{R}^n, +)$$

because $t_{\alpha-\beta} = t_\alpha (t_\beta)^{-1}$.

- It also contains the subgroup $O(n)$ that fixes the origin, $O(n) \leq \mathrm{Isom}(\mathbb{R}^n)$.

Exercises:

- Every $f \in \mathrm{Isom}(\mathbb{R}^n)$ can be written <u>uniquely</u> in the form

$$f = t_\alpha \circ \varphi$$

where $t_\alpha \in T$ and $\varphi \in O(n)$.

- The subgroup $T \leq \mathrm{Isom}(\mathbb{R}^n)$ is normal. In particular we have

$$\varphi \circ t_\alpha \circ \varphi^{-1} = t_{\varphi(\alpha)}$$