Math 562 **Spring 2014**
Homework 2 **Drew Armstrong**

Problems on Rings

1. Chinese Remainder Theorem. Given two ideals $I, J \leq R$ we define their product:

$$IJ := \langle\{ab : a \in I, b \in J\}\rangle.$$

This is the smallest ideal containing all the elements ab for $a \in I$ and $b \in J$.

(a) Prove that $IJ \leq I \cap J$.

(b) We say that ideals $I, J \leq R$ are coprime if $I + J = R$. In this case, prove that $I \cap J \leq IJ$, and hence $IJ = I \cap J$.

(c) If $I, J \leq R$ are coprime ideals, prove that the map

$$\varphi(x + IJ) := (x + I, x + J)$$

defines a ring isomorphism $R/(IJ) \approx R/I \times R/J$.

2. Groups of Units. Let R and S be rings. Prove that we have an isomorphism of groups:

$$(R \times S)^\times \approx R^\times \times S^\times.$$

3. Diamond Isomorphism for Rings. Let R be a ring, let $S \subseteq R$ be a subring, and let $I \leq R$ be an ideal.

(a) Prove that $S + I$ is a subring of R.

(b) Prove that I is an ideal of $S + I$.

(c) Prove that $S \cap I$ is an ideal of S.

(d) Prove that we have an isomorphism of rings:

$$\frac{S}{S \cap I} \approx \frac{S + I}{I}.$$

[Hint: Consider the natural map $\varphi : S \to R/I$ defined by $a \mapsto a + I$. What is the image? What is the kernel? Now use the First Isomorphism Theorem.]

Problems on Polynomials

4. Descartes' Factor Theorem. Let K be a field and consider the ring $K[x]$ of polynomials. Given $f(x) \in K[x]$ and $\alpha \in K$ such that $f(\alpha) = 0$, prove that $f(x) = (x - \alpha)h(x)$ where $h(x) \in K[x]$ with $\deg(h) = \deg(f) - 1$. [Hint: Observe that $x^n - \alpha^n = (x - \alpha)(x^{n-1} + \alpha x^{n-2} + \cdots + \alpha^{n-2}x + \alpha^{n-1})$ for all $n \geq 0$. Consider the polynomial $f(x) - f(\alpha)$.]

5. Constructing the Complex Numbers. Let $\mathbb{R}$ and $\mathbb{C}$ be the real and complex fields. Let $\varphi : \mathbb{R}[x] \to \mathbb{C}$ be the map that sends a polynomial $f(x)$ to its evaluation $f(i) \in \mathbb{C}$ at $x = i$.

(a) Prove that φ is a surjective ring homomorphism.

(b) Recall the definition of complex conjugation: $\overline{a + ib} := a - ib$ for $a, b \in \mathbb{R}$. **Prove** that $f(-i) = \overline{f(i)} \in \mathbb{C}$ for all $f(x) \in \mathbb{R}[x]$.

(c) Use Descartes' Factor Theorem to prove that the kernel of φ is the principal ideal generated by $x^2 + 1$:

$$\ker\varphi = (x^2 + 1) := \{(x^2 + 1)g(x) : g(x) \in \mathbb{R}[x]\}.$$

(d) Conclude that $\mathbb{C}$ is isomorphic to the quotient ring $\mathbb{R}[x]/(x^2 + 1)$.