Math 561 H **Fall 2011**
Exam 3 — Wed Nov 30 **Drew Armstrong**

There are 3 problems and 5 pages. This is a closed book test. Any student caught cheating will receive a score of zero.

1. Suppose that a group G acts on a set X by homomorphism $\varphi : G \to \mathrm{Aut}(X)$, and define a relation on X by

$$x \sim y \iff \exists\, g \in G \text{ such that } \varphi_g(x) = y.$$

(a) **Prove** that $\sim$ is an *equivalence* on X. (The $\sim$-classes are called G-orbits.)

Proof. For all $x \in X$ note that $x \sim x$ since $\varphi_1(x) = x$. Hence $\sim$ is **reflexive**. Next suppose that $x \sim y$; i.e. there exists $g \in G$ such that $\varphi_g(x) = y$. Then $\varphi_{g^{-1}}(y) = x$, hence $y \sim x$, so $\sim$ is **symmetric**. Finally, suppose that $x \sim y$ and $y \sim z$; i.e. there exist $g, h \in G$ such that $\varphi_g(x) = y$ and $\varphi_h(y) = z$. Then we have $\varphi_{hg}(x) = \varphi_h(\varphi_g(x)) = \varphi_h(y) = z$, hence $x \sim z$, and $\sim$ is **transitive**. □

(b) Suppose that $\varphi_g(x) = y$ (i.e. $x \sim y$). Use the group element g to **define a function** $f : \mathrm{Stab}(x) \to \mathrm{Stab}(y)$. (Hint: Conjugate by g.)

Proof. Define the map $f : \mathrm{Stab}(x) \to \mathrm{Stab}(y)$ by $f(h) := ghg^{-1}$, and note that if $h \in \mathrm{Stab}(x)$ — i.e. $\varphi_h(x) = x$ — then indeed $f(h) = ghg^{-1} \in \mathrm{Stab}(y)$ since

$$\varphi_{ghg^{-1}}(y) = \varphi_g(\varphi_h(\varphi_{g^{-1}}(y))) = \varphi_g(\varphi_h(x)) = \varphi_g(x) = y.$$

□

(c) Prove that f is **bijection**.

Proof. Note that the map $\psi(h) := g^{-1}hg$ maps $\mathrm{Stab}(y) \to \mathrm{Stab}(x)$ and satisfies $f \circ \psi = \psi \circ f = 1$. Hence $f^{-1} = \psi$ and f is a bijection. □

(d) Prove that f is a **homomorphism**, hence $\mathrm{Stab}(x) \approx \mathrm{Stab}(y)$.

Proof. Given $h, k \in \mathrm{Stab}(x)$, note that

$$f(h)f(k) = (ghg^{-1})(gkg^{-1}) = g(hk)g^{-1} = f(hk).$$

□

2. For $\alpha \in \mathbb{R}^n$ define the translation $t_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ by $t_\alpha(x) := x + \alpha$, and consider the group $\mathrm{GL}(\mathbb{R}^n)$ of invertible linear maps $\varphi : \mathbb{R}^n \to \mathbb{R}^n$.

(a) For all $\alpha \in \mathbb{R}^n$ and $\varphi \in \mathrm{GL}(\mathbb{R}^n)$, **prove that** $\varphi \circ t_\alpha = t_{\varphi(\alpha)} \circ \varphi$.

Proof. For all $x \in \mathbb{R}^n$ we have

$$\varphi \circ t_\alpha(x) = \varphi(t_\alpha(x)) = \varphi(x + \alpha) = \varphi(x) + \varphi(\alpha) = t_{\varphi(\alpha)}(\varphi(x)) = t_{\varphi(\alpha)} \circ \varphi(x).$$

□

(b) Let $\mathbb{R}^n_+ := \{t_\alpha : \alpha \in \mathbb{R}^n\}$ be the group of translations of $\mathbb{R}^n$, and let

$$\mathrm{Aff}(\mathbb{R}^n) := \{t_\alpha \circ \varphi : \ t_\alpha \in \mathbb{R}^n_+, \ \varphi \in \mathrm{GL}(\mathbb{R}^n)\}.$$

Use part (a) to **verify that** $\mathrm{Aff}(\mathbb{R}^n)$ **is a group**.

Proof. Let $t_0 \in \mathbb{R}^n_+$ be translation by the zero vector and let $I \in \mathrm{GL}(\mathbb{R}^n)$ be the identity linear map. Then $t_0 \circ I \in \mathrm{Aff}(\mathbb{R}^n)$ is the identity map on $\mathbb{R}^n$. Now consider an arbitrary element $t_\alpha \circ \varphi \in \mathrm{Aff}(\mathbb{R}^n)$ and observe that its inverse satisfies

$$(t_\alpha \circ \varphi)^{-1} = \varphi^{-1} \circ t_\alpha^{-1} = \varphi^{-1} \circ t_{-\alpha} = t_{\varphi^{-1}(-\alpha)} \circ \varphi^{-1} \in \mathrm{Aff}(\mathbb{R}^n).$$

Finally, consider $t_\alpha \circ \varphi$ and $t_\beta \circ \mu$ in $\mathrm{Aff}(\mathbb{R}^n)$ and note that

$$(t_\alpha \circ \varphi) \circ (t_\beta \circ \varphi) = t_\alpha \circ t_{\varphi(\beta)} \circ \varphi \circ \mu = t_{\alpha+\varphi(\beta)} \circ (\varphi \circ \mu) \in \mathrm{Aff}(\mathbb{R}^n).$$

□

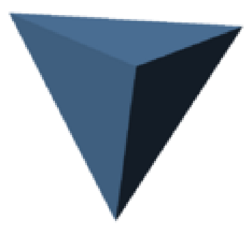
(c) Use part (a) to **prove that** $\mathbb{R}^n_+ \trianglelefteq \mathrm{Aff}(\mathbb{R}^n)$.

Proof. Consider an arbitrary element $t_\alpha \in \mathbb{R}^n_+$ and an arbitrary element $t_\beta \circ \varphi \in \mathrm{Aff}(\mathbb{R}^n)$. Then we have

$$\begin{aligned}(t_\beta \circ \varphi) \circ t_\alpha \circ (t_\beta \circ \varphi)^{-1} &= t_\beta \circ \varphi \circ t_\alpha \circ \varphi^{-1} \circ t_{-\beta} \\ &= t_\beta \circ t_{\varphi(\alpha)} \circ (\varphi \circ \varphi^{-1}) \circ t_{-\beta} \\ &= t_\beta \circ t_{\varphi(\alpha)} \circ t_{-\beta} \\ &= t_{\varphi(\alpha)} \in \mathbb{R}^n_+.\end{aligned}$$

□

3. Let T be the group of **rotational** symmetries of a regular tetrahedron, shown below.



(a) T acts transitively on the set of 4 vertices. Use Orbit-Stabilizer to compute $|T|$.

Proof. Let v be a vertex, so $|\mathrm{Orb}(v)| = 4$. Note that $\mathrm{Stab}(v)$ is a cyclic group of size 3. Hence $|T| = |\mathrm{Orb}(v)||\mathrm{Stab}(v)| = 4 \cdot 3 = 12$. □

(b) Let $N \trianglelefteq T$ be a **normal** subgroup. Based on Lagrange's Theorem, what are the possible sizes of N?

Proof. Lagrange's Theorem says that $|N|$ divides $|T| = 12$. Hence $|N|$ is in the set $\{1, 2, 3, 4, 6, 12\}$. □

(c) The group T has 3 conjugacy classes. **List their sizes**.

[I'm very sorry. There are actually 4 conjugacy classes. I realize that this error could have thrown people off the trail and so I graded Problem 3 very carefully. Fortunately, this difference doesn't affect parts (d), (e), (f), (g) very much. Also fortunately (or maybe unfortunately), it didn't seem to matter much — people who missed this problem tended to have bigger issues.]

Proof. I would have accepted $1, 3, 8$ or $1, 3, 4, 4$ as completely correct. Because: As always, the identity is its own conjugacy class or size 1. There is one rotation around each of the 6 edges, but the same rotation is shared by a pair of opposite edges, hence this class has size 3. There are two non-identity rotations about each vertex (or its opposite face), for a total of 8 elements. I naively assumed these formed a class of size 8, but actually the rotations by $2\pi/3$ and $-2\pi/3$ about a vertex are not conjugate in T, so we get two classes of size 4. Check: $1+3+4+4=12$. $\square$

(d) Again let $N \trianglelefteq T$. Based on parts (b) and (c), which values of $|N|$ are possible?

Proof. Since N is closed under conjugation, it is a union of conjugacy classes — and since $1 \in N$, one of these classes must be the identity class. By part (c), $|N|$ is equal to 1 plus numbers from $\{3,4,4\}$. We conclude that $|N| \in \{1,4,5,8,9,12\}$. Combining with part (b) yields $|N| \in \{1,4,12\}$. [If T is not simple, then it must have a normal subgroup of size 4. In fact it does, but you don't need to show this.] $\square$

(e) Now let T act on the set F of four faces of the tetrahedron. Each $g \in T$ partitions F into "cycles" — which are the orbits of $\langle g \rangle$ acting on F — and the number of cycles is constant for g in a given conjugacy class of T. **For each of the 4 conjugacy classes of T, list the number of associated cycles**.

(f) Now we will color the faces of the tetrahedron using at most k colors. There are k^4 colorings if the tetrahedron is not allowed to move. **For each of the 4 conjugacy classes of T, list the number of colorings that are fixed by an element of the class**.

I will collect the answers (c), (e) and (f) in the following table.

rotate around	size of class	number of cycles	number of fixed colorings
	1	4	k^4
edge	3	2	k^2
vertex/face	4	2	k^2
vertex/face	4	2	k^2

(g) Finally, use Burnside's Lemma to **compute the number of T-orbits of face colorings with at most k-colors**. (Hint: When $k=2$ the answer is 5.)

Proof. By Burnside's Lemma, the number of orbits of colorings is the average number of colorings fixed by an element of T. Using the data in the table, the number of orbits is

$$\frac{1}{12}[k^4+3k^2+4k^2+4k^2] = \frac{k^4+11k^2}{12}.$$

Check: Putting $k=2$ we obtain $(16+11\cdot 4)/12 = 5$ black-and-white colorings of the tetrahedron. In fact, there is (up to rotation) exactly one way to color the tetrahedron with i black faces and $4-i$ white faces, for $i=0,1,2,3,4$. $\square$