Math 561 H **Fall 2011**
Exam 1 — Wed Sep 28 **Drew Armstrong**

There are 4 problems and 4 pages. This is a closed book test. Any student caught cheating will receive a score of zero. The problems are (mostly) cumulative. In any problem, you may assume the results from earlier problems.

1. Let $\varphi : G \to H$ be a map between groups. **Define** what it means for φ to be

(a) [**2 points**] an injection,

$$\boxed{\forall a, b \in G, \varphi(a) = \varphi(b) \Rightarrow a = b}$$

(b) [**2 points**] a surjection,

$$\boxed{\forall h \in H, \exists g \in G, \varphi(g) = h}$$

(c) [**2 points**] a homomorphism,

$$\boxed{\forall a, b \in G, \varphi(ab) = \varphi(a)\varphi(b)}$$

(d) [**2 points**] an isomorphism.

$$\boxed{\varphi \text{ is an injection, a surjection, and a homomorphism}}$$

2. If $\varphi : G \to H$ is a homomorphism, **prove** that

(a) [**2 points**] $\varphi(1_G) = 1_H$,

Proof. Note that $1_G 1_G = 1_G$ and apply φ to get $\varphi(1_G)\varphi(1_G) = \varphi(1_G)$. Then multiply both sides (say on the left) by $\varphi(1_G)^{-1}$ to get $\varphi(1_G) = 1_H$. □

(b) [**2 points**] $\varphi(a^{-1}) = \varphi(a)^{-1}$ for all $a \in G$,

Proof. Note that $aa^{-1} = 1_G$ and apply φ to get $\varphi(a)\varphi(a^{-1}) = \varphi(1_G)$. By part (a) this implies $\varphi(a)\varphi(a^{-1}) = 1_H$. Now multiply both sides on the left by $\varphi(a)^{-1}$ to get $\varphi(a^{-1}) = \varphi(a)^{-1}$. □

(c) [**3 points**] $\ker\varphi$ is a subgroup of G.

Proof. To show that $\ker\varphi$ is closed, let $a, b \in \ker\varphi$. That is, we have $\varphi(a) = \varphi(b) = 1_H$. Then since φ is a homomorphism we have $\varphi(ab) = \varphi(a)\varphi(b) = 1_H 1_H = 1_H$, hence $ab \in \ker\varphi$. To show that $\ker\varphi$ is closed under inversion, let $a \in \ker\varphi$. Then by part (b) we have $\varphi(a^{-1}) = \varphi(a)^{-1} = 1_H^{-1} = 1_H$, hence $a^{-1} \in \ker\varphi$. Finally, part (a) implies that $\ker\varphi$ contains the identity 1_G. □

3. If $\varphi : G \to H$ is a homomorphism, **prove** that

(a) [**3 points**] if φ is injective then $\ker\varphi = \{1_G\}$,

Proof. Suppose that φ is injective and let $a \in \ker\varphi$. Then $\varphi(a) = 1_H = \varphi(1_G)$ by **2.**(a). Then injectivity implies $a = 1_G$. Hence $\ker\varphi = \{1_G\}$. □

(b) [**4 points**] if $\ker\varphi = \{1_G\}$ then φ is injective.

Proof. Suppose that $\ker\varphi = \{1_G\}$ and let $\varphi(a) = \varphi(b)$. Then by **2.**(b) we have $\varphi(a^{-1}b) = \varphi(a^{-1})\varphi(b) = \varphi(a)^{-1}\varphi(b) = \varphi(b)^{-1}\varphi(b) = 1_H$, hence $a^{-1}b \in \ker\varphi$. Since $\ker\varphi = \{1_G\}$ this implies that $a^{-1}b = 1_G$, or $a = b$. Hence φ is injective. $\square$

4. Let H be a subgroup of G. Define a relation on G by setting $a \sim b \Leftrightarrow a^{-1}b \in H$.

(a) [**3 points**] **Prove** that $\sim$ is an equivalence relation.

Proof. To show transitivity, let $a \sim b$ and $b \sim c$. That is, we have $a^{-1}b \in H$ and $b^{-1}c \in H$. Since H is closed under the group operation this implies $a^{-1}c = (a^{-1}b)(b^{-1}c) \in H$, hence $a \sim c$ as desired. To show symmetry, let $a \sim b$. That is, we have $a^{-1}b \in H$. Since H is closed under inversion this implies that $b^{-1}a = (a^{-1}b)^{-1} \in H$, hence $b \sim a$ as desired. Finally, note that $a^{-1}a = 1_G \in H$ and hence $a \sim a$ for all $a \in G$. $\square$

(b) [**3 points**] **Prove** that there exists a bijection between any two $\sim$-classes.

Proof. Given $a, b \in G$, consider the equivalence classes $[a] = \{g \in G : g^{-1}a \in H\}$ and $[b] = \{g \in G : g^{-1}b \in H\}$. I claim that the map $f : G \to G$ defined by $f(g) = ba^{-1}g$ restricts to a bijection $f : [a] \to [b]$. First suppose that $g \in [a]$, i.e. $g^{-1}a \in H$. Then $f(g)^{-1}b = (ba^{-1}g)^{-1}b = g^{-1}a \in H$, hence $f(g) \in [b]$. That is, f really does map $[a]$ into $[b]$. To show that f is injective, suppose that $f(g) = f(h)$ for some $g, h \in [a]$. That is, $ba^{-1}g = ba^{-1}h$. Multiply both sides on the left by ab^{-1} to get $g = h$. Finally, to show that f is surjective, consider any $h \in [b]$ (i.e. $h^{-1}b \in H$) and note that $f(ab^{-1}h) = h$. We are done if we can show that $ab^{-1}h \in [a]$. But this is true because $(ab^{-1}h)^{-1}a = h^{-1}b \in H$. $\square$