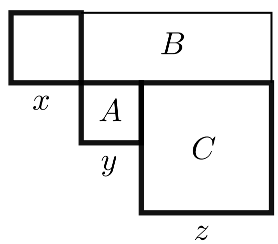
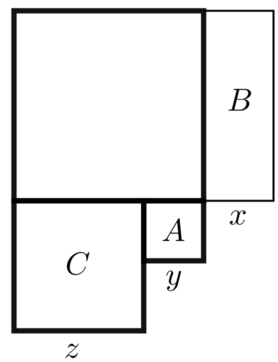
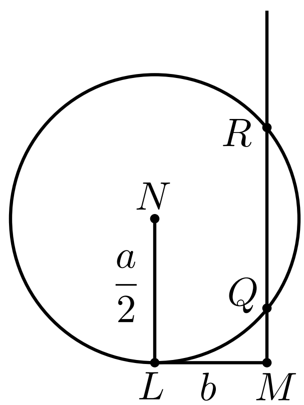
Math 461 **Spring 2020**
Homework 1 **Drew Armstrong**

Problem 1. Al-Khwarizmi. If we do not accept negative numbers, then the hardest type quadratic equation comes down to the following geometric problem. In each diagram we have three squares and one rectangle. Assuming that the lengths satisfy $x + y = z$, prove that the areas satisfy $A + B = C$.





Problem 2. Descartes. Consider the following figure from Descartes' *Geometry* (1637):



(a) Prove that the distances MQ and MR are solutions to the equation $y^2 + b^2 = ay$. [Hint: Let L be at the origin of the x, y-plane and use the equation of a circle.]
(b) Explain how the discriminant of the polynomial $y^2 - ay + b^2$ is related to the picture.

Problem 3. Same Function $\Rightarrow$ Same Coefficients. Let $f(x)$ and $g(x)$ be polynomials and suppose that we have $f(a) = g(a)$ for all real numbers a. In this case prove that $f(x)$ and $g(x)$ have exactly the same coefficients. [Hint: Consider the polynomial $h(x) = f(x) - g(x)$. If $h(x)$ has at least one nonzero coefficient then we proved in class that the equation $h(x) = 0$ has finitely many solutions.]

Problem 4. Discriminant of a Quadratic. Suppose that the quadratic polynomial $f(x) = x^2 + px + q = 0$ can be factored as $x^2 + px + q = (x - r)(x - s)$ for some real numbers r and s.

(a) Use Problem 3 to show that $p = -r - s$ and $q = rs$.
(b) Show that $\text{Disc}(f) = (r - s)^2$.

(c) Show that $\mathrm{Disc}(f) = 0$ if and only if $r = s$.

Problem 5. Factoring Cubic Polynomials. Factor each of the following cubic polynomials as $f(x) = (x-r)(x-s)(x-t)$ by first guessing a root, then using long division, and finally using the quadratic formula.

(a) $f(x) = x^3 - 2x^2 - 5x + 6$
(b) $f(x) = x^3 - 3x^2 + x + 1$
(c) $f(x) = x^3 - 1$

Problem 6. Alternate Proof of Descartes' Factor Theorem.

(a) For any variable x, constant a and positive integer n, show that

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}).$$

(b) For any polynomial $f(x)$ and constant a, use part (a) to show that

$$f(x) - f(a) = (x-a)g(x)$$

for some polynomial $g(x)$.