In class we proved that $\sqrt{2}$ is not a rational number (the edge and diagonal of a square are incommensurable). This "crisis of incommensurables" forced the Greeks to base their mathematics on the concept of "length" instead of "number". In particular, the Euclidean system was based on constructions with straightedge-and-compass, which are restricted to

- drawing the line containing two given points,
- drawing the circle with a given center and radius,
- drawing the intersection points of lines and circles.

(Why did they do this? Well, you can't prove anything if you don't have some rules.) However, this system ran into trouble. Classical mathematics was unable to solve the following problems, and it was not for lack of trying:

(1) to draw a square with area equal to a given circle,
(2) to draw the edge of a cube with volume **double** that of a given cube,
(3) to draw a line that **trisects** the angle between two given lines,
(4) to draw the regular heptagon.

Many people suspected that these problems were impossible, but no one could prove it until Fermat and Descartes (*La Géométrie*, 1637) finally healed the rift between geometry and algebra by introducing **coordinate geometry**. In this note, I will use coordinate geometry to prove that problems (2),(3),(4) are impossible; Lindemann (1882) proved that (1) is impossible, but his proof involves too many integrals for this class.

The main tool we will use is the following lemma on quadratic field extensions.

Lemma. *Consider a quadratic field extension $\mathbb{F} \subseteq \mathbb{F}[\sqrt{c}]$ and a* **cubic** *polynomial $f(x)$ with coefficients in $\mathbb{F}$. If $f(x)$ has a root in $\mathbb{F}[\sqrt{c}]$, then it also has a root in F.*

Proof. Suppose we have $f(\alpha) = 0$ with $\alpha \in \mathbb{F}[\sqrt{c}]$. If $\alpha \in \mathbb{F}$ we are done. Otherwise, let $(a + b\sqrt{c})^* = a - b\sqrt{c}$ be the conjugation map of the extension and note that

$$f(\alpha^*) = f(\alpha)^* = 0^* = 0.$$

Hence $\alpha^* \neq \alpha$ is **another** root of $f(x)$. Using Descartes' Factor Theorem we can write

$$f(x) = a(x - \alpha)(x - \alpha^*)(x - \beta),$$

where β is in $\mathbb{F}[\sqrt{c}]$. We claim in fact that $\beta \in \mathbb{F}$. If not, then β^* is another solution to $f(x) = 0$, not equal to α, α^*, β, which is impossible because f has degree 3. $\square$

Now we can prove the main result.

Theorem. *Let $f(x) \in \mathbb{Q}[x]$ be a* **cubic** *polynomial with rational coefficients, but* **no** *rational roots. Then the roots of $f(x)$ are not constructible with straightedge-and-compass.*

Proof. Suppose, for contradiction, that $f(\alpha) = 0$ for some constructible α. Then there exists a chain of quadratic extensions

$$\mathbb{Q} = \mathbb{F}_0 \subseteq \mathbb{F}_1 \subseteq \mathbb{F}_2 \subseteq \cdots \subseteq \mathbb{F}_k,$$

such that $\alpha \in \mathbb{F}_k$. Since the coefficients of $f(x)$ are in $\mathbb{Q} \subseteq \mathbb{F}_{k-1}$, the lemma implies that $f(x)$ has a root in $\mathbb{F}_{k-1}$. By repeatedly applying the lemma, we conclude that $f(x)$ has a root in $\mathbb{F}_0 = \mathbb{Q}$, which is a contradiction. $\square$

But how does this theorem apply to the classical construction problems? It turns out that each (except (1)) is governed by a certain cubic polynomial with rational coefficients.

(2) **Doubling the Cube.** If any cube can be doubled, then the unit cube can be doubled, which implies that the number $\sqrt[3]{2} \in \mathbb{R}$ is constructible. But $\sqrt[3]{2}$ is a root of the cubic polynomial $x^3 - 2 \in \mathbb{Q}[x]$, and I claim that this polynomial has no rational root. Indeed, suppose that $x = a/b$ is a root with $a, b \in \mathbb{Z}$ and a, b coprime. Then we have

$$\begin{aligned} \frac{a^3}{b^3} - 2 &= 0, \\ a^3 - 2b^3 &= 0, \\ a^3 &= 2b^3. \end{aligned}$$

Since a divides $2b^3$ and a is coprime to b we conclude that a divides 2. Also, since b divides a^3, we conclude that $b = \pm 1$. Thus the only possibilities for a/b are

$$\frac{a}{b} = \frac{\pm 1, 2}{\pm 1} = \pm 1, \pm 2.$$

We can plug in the numbers $\pm 1, \pm 2$ to see that none of them is a root of $x^3 - 2$. Hence this polynomial has no rational root, and the theorem implies that $\sqrt[3]{2} \in \mathbb{R}$ is **not** constructible. Hence it is impossible to "double the cube".

(3) **Trisecting the Angle.** If an arbitrary angle can be trisected, then the angle $\pi/3$ can be trisected, in which case the number $\cos(\pi/9) \in \mathbb{R}$ is constructible. But putting $\theta = \pi/9$ into the identity $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$ yields

$$4\cos^3\left(\frac{\pi}{9}\right) - 3\cos\left(\frac{\pi}{9}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2},$$

and then setting $u = 2\cos(\pi/9)$ gives us a cubic polynomial with rational coefficients:

$$\begin{aligned} 4\frac{u^3}{8} - 3\frac{u}{2} - \frac{1}{2} &= 0, \\ u^3 - 3u - 1 &= 0. \end{aligned}$$

Using the above method, we see that the only possible rational roots are $u = a/b$ with $a/b = \pm 1$, and neither of these is a root. Hence the polynomial $u^3 - 3u - 1 \in \mathbb{Q}[u]$ has no rational root and the theorem implies that $2\cos(\pi/9)$, and hence $\cos(\pi/9)$, is not constructible. We conclude that it is impossible to "trisect the angle".

(4) **Constructing the Regular Heptagon.** If the regular 7-gon can be constructed, then the regular 7-gon centered at the origin and containing the point $(1, 0)$ can be constructed, which implies that the point $(\cos(2\pi/7), \sin(2\pi/7))$ can be constructed. Let $\omega = e^{2\pi i/7}$ and note that $2\cos(2\pi/7) = u = \omega + \omega^{-1}$ satisfies

$$u^3 + u^2 - 2u - 1 = \omega^3 + \omega^2 + \omega + 1 + \omega^{-1} + \omega^{-2} + \omega^{-1} = 0.$$

But one can show again that the cubic polynomial $x^3 + x^2 - 2x - 1 \in \mathbb{Q}[x]$ has no rational root. This implies that $2\cos(2\pi/7)$, and hence $\cos(2\pi/7)$ is not constructible. Hence it is impossible to construct the regular heptagon.

These three theorems were first rigorously proved by Pierre Wantzel in 1837. His proof was similar to ours but more general, because he actually proved more. Wantzel's name is remembered by the "Gauss-Wantzel Theorem", which states that the regular n-gon is constructible with straightedge-and-compass if and only if $\varphi(n)$ is a power of 2.