2/16/15

Exam 1 is Wednesday in class.
(Discuss cheating policy.)

Today: Review for Exam 1.

I have handed out the exam I gave four years ago. Problems 3 and 4 are not completely relevant because I am taking a different path this semester.

In particular, I am telling you more about rings & fields than I told them. Your Exam 1 will match the material in the course notes and homeworks 1 & 2.

Topics:

- The Quadratic Formula
  — "completing the square"
- Interpreting the discriminant b^2-4ac.

- The ring of polynomials $\mathbb{F}[x]$.
- Polynomials as formal expressions vs. polynomials as functions.

- The Division Theorem:

Given $f(x), g(x) \in \mathbb{F}[x]$ with $g(x) \neq 0$, there exist unique $q(x), r(x) \in \mathbb{F}[x]$ such that

- $f(x) = q(x)g(x) + r(x)$
- $r(x) = 0$ or $\deg(r) < \deg(g)$.

- Descartes' Factor Theorem:

Given $f(x) \in \mathbb{F}[x]$ and $\alpha \in \mathbb{F}$ we have

$$f(\alpha) = 0 \iff (x-\alpha) \text{ divides } f(x)$$
$$(\text{i.e. } \exists\, g(x) \in \mathbb{F}[x],\ f(x) = (x-\alpha)g(x)).$$

Know how to prove this using the Division Theorem.

- Use DFT to prove that a polynomial $f(x) \in \mathbb{F}[x]$ of degree n has at most n distinct roots in $\mathbb{F}$.

- If $f(x) \in \mathbb{F}[x]$ has degree n and has n distinct roots, $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{F}$, use DFT to show that

$$f(x) = a(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$$

where a is the leading coefficient.
[See HW 2.4.]

- Be able to apply the above theorems to concrete situations

- The general cubic equation

$$ax^3 + bx^2 + cx + d = 0$$

can be depressed by substituting $x = y - b/3a$ to obtain

$$y^3 + py + q = 0,$$

where p & q are complicated expressions in a, b, c, d.

- The depressed cubic $y^3 + py + q = 0$ can be solved with a <u>trick</u>:

If $y = u + v$ then we have

$$y^3 - 3uv\,y - (u^3 + v^3) = 0.$$

We know $p = -3uv$ and $q = -(u^3 + v^3)$ and we want u & v. Instead we solve for u^3 & v^3 using

$$\begin{aligned}(z - u^3)(z - v^3) &= z^2 - (u^3 + v^3)z + u^3v^3 \\ &= z^2 + qz - p^3/27.\end{aligned}$$

The QF gives

$$\begin{aligned}u^3, v^3 &= \frac{-q \pm \sqrt{q^2 + 4p^3/27}}{2} \\ &= -\left(\frac{q}{2}\right) \pm \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}\end{aligned}$$

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$\downarrow$

- Cardano's Formula says

$y = u + v$

$$= \sqrt[3]{-\left(\frac{q}{2}\right) + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\left(\frac{q}{2}\right) + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$$

- Careful consideration of Cardano's Formula led to the acceptance of complex numbers.

- Know the three ways to think of $\mathbb{C}$:

  - as numbers $a+ib$
  - as vectors (a, b)
  - as linear functions $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

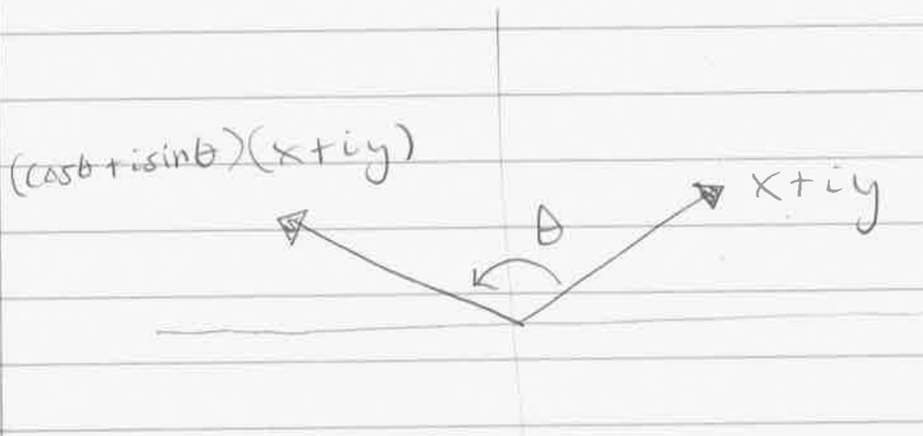
- Know how to <u>rotate</u> using complex numbers:

To rotate $x+iy$ counterclockwise by angle θ, <u>multiply</u> by $\cos\theta + i\sin\theta$.

$(\cos\theta + i\sin\theta)(x+iy) =$

$$(x\cos\theta - y\sin\theta) + i(x\sin\theta + y\cos\theta).$$

Picture:



- Use this idea to prove de Moivre's Theorem:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

"rotate n times by θ = rotate once by $n\cdot\theta$".

Example: $n=2$.

$$\cos(2\theta) + i\sin(2\theta) = (\cos\theta + i\sin\theta)^2$$

$$= (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)$$

$$= \cos^2\theta + i\sin\theta\cos\theta + i\sin\theta\cos\theta + i^2\sin^2\theta.$$

$$= (\cos^2\theta - \sin^2\theta) + i(2\sin\theta\cos\theta).$$

Equating real and imaginary parts gives

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$\sin(2\theta) = 2\sin\theta\cos\theta \quad .$$

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