Math 461 **Spring 2015**
Exam 2 (Wed Mar 25) **Drew Armstrong**

There are 3 problems with 12 parts. Each part is worth 2 points for a total of 24 points. If two exams are submitted with copied answers then **both** exams will receive 0 points.

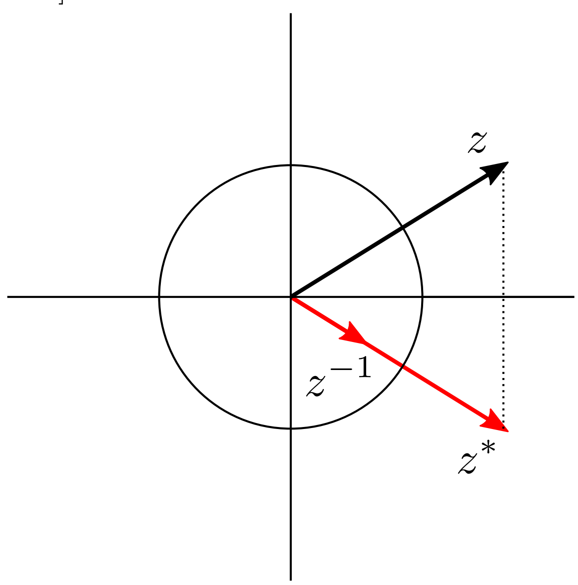
1. Complex Numbers.

(a) Let $z \in \mathbb{C}$. Write a formula for z^{-1} in terms of the complex conjugate z^*.

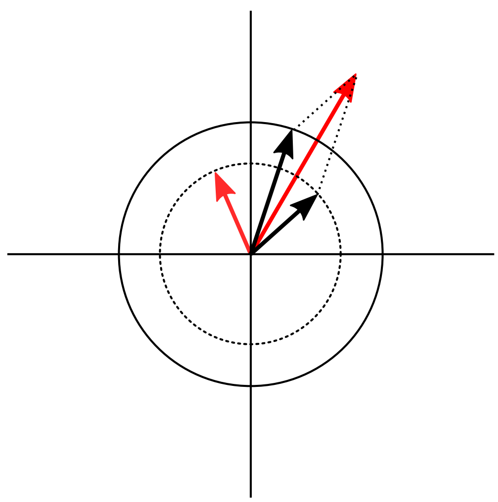
We "rationalize the denominator" to get

$$z^{-1} = \frac{1}{z} = \frac{1}{z} \cdot \frac{z^*}{z^*} = \frac{z^*}{zz^*} = \frac{z^*}{|z|^2}.$$

(b) Draw the **complex conjugate** and the **inverse** of the given complex number. [Hint: The unit circle is shown.]



(c) Draw the **sum** and **product** of the two given complex numbers. [Hint: The unit circle is shown.]



2. De Moivre's Formula.

(a) Accurately state de Moivre's Formula.

For all integers n and real numbers θ we have

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

(b) Use de Moivre's Formula to express $\cos(2\theta)$ as a function of $\cos(\theta)$.

De Moivre's Formula says

$$\begin{aligned}\cos(2\theta) + i\sin(2\theta) &= (\cos\theta + i\sin\theta)^2 \\ &= \cos^2\theta + 2i\sin\theta\cos\theta + i^2\sin^2\theta \\ &= (\cos^2\theta - \sin^2\theta) + i(2\sin\theta\cos\theta).\end{aligned}$$

Then comparing real parts gives

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= \cos^2\theta - (1 - \cos^2\theta) \\ &= 2\cos^2\theta - 1.\end{aligned}$$

(c) Given the fact that $\cos(2\pi/5) = \frac{-1+\sqrt{5}}{4}$, use part (b) to compute the value of $\cos(4\pi/5)$.

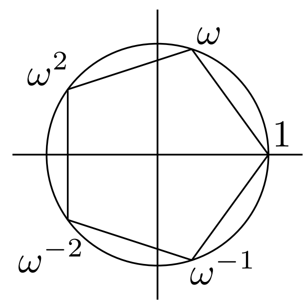
Substituting $\theta = 2\pi/5$ into the formula from part (b) gives

$$\begin{aligned}\cos(4\pi/5) &= 2\cos^2(2\pi/5) - 1 \\ &= 2\left(\frac{-1+\sqrt{5}}{4}\right)^2 - 1 \\ &= 2\left(\frac{1 - 2\sqrt{5} + 5}{16}\right) - 1 \\ &= \frac{6 - 2\sqrt{5}}{8} - \frac{8}{8} \\ &= \frac{-2 - 2\sqrt{5}}{8} \\ &= \frac{-1 - \sqrt{5}}{4}.\end{aligned}$$

3. Roots of Unity. Throughout this problem, let $\omega = e^{2\pi i/5}$.

(a) Label the vertices of the following regular pentagon with powers of ω. [Hint: The unit circle is shown.]

There are infinitely many possible answers. Here is one.



(b) Factor $x^5 - 1$ over the **complex numbers**.

$$x^5 - 1 = (x-1)(x-\omega)(x-\omega^{-1})(x-\omega^2)(x-\omega^{-2})$$

(c) List the **primitive** 5th roots of unity.

The primitive 5th roots of unity are $\omega, \omega^{-1}, \omega^2, \omega^{-2}$.

(d) Factor $1 + x + x^2 + x^3 + x^4$ over the **complex numbers**.

$$1 + x + x^2 + x^3 + x^4 = (x-\omega)(x-\omega^{-1})(x-\omega^2)(x-\omega^{-2})$$

(e) Expand the product $(x-\omega)(x-\omega^{-1})$. [Hint: $\cos(2\pi/5) = (-1+\sqrt{5})/4$.]

We have

$$\begin{aligned}
(x-\omega)(x-\omega^{-1}) &= x^2 - (\omega + \omega^{-1})x + \omega\omega^{-1} \\
&= x^2 - 2\cos(2\pi/5)x + 1 \\
&= x^2 - 2\left(\frac{-1-\sqrt{5}}{4}\right)x + 1 \\
&= x^2 - \frac{-1-\sqrt{5}}{2}x + 1.
\end{aligned}$$

(f) Factor $1 + x + x^2 + x^3 + x^4$ over the **real numbers**. Your final answer should not involve sines or cosines.

From 3(d), 3(e), and 2(c), we have

$$\begin{aligned}
1 + x + x^2 + x^3 + x^4 &= (x-\omega)(x-\omega^{-1})(x-\omega^2)(x-\omega^{-2}) \\
&= (x^2 - (\omega+\omega^{-1})x + \omega\omega^{-1})(x^2 - (\omega^2+\omega^{-2})x + \omega^2\omega^{-2}) \\
&= (x^2 - 2\cos(2\pi/5)x + 1)(x^2 - 2\cos(4\pi/5)x + 1) \\
&= \left(x^2 - \frac{-1-\sqrt{5}}{2}x + 1\right)\left(x^2 - \frac{-1+\sqrt{5}}{2}x + 1\right).
\end{aligned}$$