HW 5 due Fri

G.U.

Today & Tom. 2-30-4:00.

MATH CLUB TODAY 5 p.m.!

Now, "Partial Fractions".

We know how to add fractions

\[ \frac{r}{p} + \frac{s}{q} = \frac{qr + ps}{pq} \]

But can we "UN-add" them?

E.g. \[ \frac{8}{15} = \frac{?}{p} + \frac{?}{q} \] where \( pq = 15 \)

Yes! If we can factor the denominator 15.

15 = (3 \cdot 5) prime decomposition.

\[ \frac{8}{15} = \frac{?}{3} + \frac{?}{5} \]
Method: Since 3, 5 are primes we can find \( x, y \) s.t. \( x3 + y5 = 1 \).

Do it:

\[
\begin{align*}
\frac{5}{3} &= 1 \cdot \frac{5}{3} + 2 \\
\frac{2}{5} &= 1 \cdot \frac{2}{5} + \left( \frac{1}{3} \right)
\end{align*}
\]

\( \gcd(3, 5) = 1 \).

\[ \gcd(3, 5) = 1 = \frac{3}{3} - 1 \cdot \frac{2}{5} = \frac{3}{3} - 1 \cdot \frac{2 \cdot 3 - 1 \cdot 5}{15} \]

\[ = 2 \cdot 3 - 1 \cdot 5. \]

\( \text{Not unique.} \)

\[
\begin{pmatrix}
(2+5k)3 + (-1+3k)5 \\
= (2 \cdot 3 - 1 \cdot 5) + 3 \cdot 5k - 3 \cdot 3k
\end{pmatrix}
= 1 \quad \text{for any } k \in \mathbb{Z}.
\]

Divide by 15 to get:

\[
\frac{1}{15} = \frac{2 \cdot 3}{15} - \frac{1 \cdot 5}{15} = \frac{2}{5} - \frac{1}{3}.
\]
Multiply by 8 to get

\[ \frac{8}{15} = \frac{16}{5} - \frac{8}{3} \]

\[ \checkmark \]

a "partial fraction" decomposition.

We can do the same for polynomials.

Given \( p(x), q(x) \in \mathbb{F}[x] \) we say

\[ \frac{p(x)}{q(x)} \] is a rational function of \( x \).

Eg. \( \frac{3x^2 + 7}{x^3 - x^2 + 4x - 4} \in \mathbb{Q}(x) \)

\[ \text{found brackets for ratio fractions.} \]

Theorem (Partial Fractions).

Let \( \frac{p(x)}{q(x)} \in \mathbb{F}(x) \) and suppose:

\[ q(x) = q_1(x)^n q_2(x)^m_2 \ldots q_k(x)^m_k \]

is the prime decomps. of \( q(x) \in \mathbb{F}[x] \).
Then \( \exists! \) (there exists unique) poly.

\[ p(x)/q(x) = \frac{b(x)}{q_1(x)} \]

where \( \deg a_{ij} < \deg q_i \) or \( a_{ij} = 0 \) \( \forall i,j \).

Proof? - just a sketch.

it uses Euclidean Algorithm.

Let \[ \frac{p(x)}{q(x)} = \frac{p(x)}{q_1(x)q_2(x)} \]

where \( q_1(x), q_2(x) \) are "prime".
i.e. \( \text{gcd}(\psi_1(x), \psi_2(x)) = 1 \) so.

3. \( A(x), B(x) \in \mathbb{F}[x] \) s.t.

\[ 1 = A(x) \psi_1(x) + B(x) \psi_2(x). \]

Divide by \( g(x) = \psi_1(x) \psi_2(x) + 7x + 1 \):

\[ \frac{1}{\psi_1 \psi_2} = \frac{A \psi_1}{\psi_1 \psi_2} - \frac{B \psi_2}{\psi_1 \psi_2} \]

\[ \frac{1}{g} = \frac{A}{\psi_2} + \frac{B}{\psi_1} \]

\[ \frac{p(x)}{g(x)} = \frac{p(x) A(x)}{\psi_2(x)} + \frac{p(x) B(x)}{\psi_1(x)} \]

then use long division:

\[ = R(x) + \frac{\alpha_1(x)}{\psi_1(x)} + \frac{\alpha_2(x)}{\psi_2(x)}. \]
eg. \[ \frac{3x^2 + 7}{x^3 - x^2 + 4x - 4}. \]

**Step 1:** Factor the denominator. (Sometimes HARD!)

\[ x^3 - x^2 + 4x - 4 = (x - 1)(x^2 + 4). \]

Both prime over \( \mathbb{Q} \).

**Step 2:** Theorem says \( \exists ! A, B, C \in \mathbb{Q} \) s.t.

\[ \frac{3x^2 + 7}{(x - 1)(x^2 + 4)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 4}. \]

**Step 3:** Find \( A, B, C \).

Equate numerators

\[ 3x^2 + 7 = A(x^2 + 4) + (Bx + C)(x - 1). \]

Plug \( x = 1 \) to get

\[ 3 + 7 = A(5) + 0. \]

\[ 10 = 5A \implies A = 2. \]
So... $3x^2 + 7 = 2(x^2+c) + (Bx+C)(x-1)$

\[= 2x^2 + 8 + Bx^2 +Cx - Bx - C.\]

$3x^2 - 6x + 7 = (2+B)x^2 + (C-B)x + (8-C)$

Equate coefficients

\[3 = 2 + B \quad \Rightarrow \quad B = 1\]
\[0 = B - C \quad \Rightarrow \quad C = -1\]
\[7 = 8 - C\]

Conclusion.

\[\frac{3x^2 + 7}{(x-1)(x^2+c)} = \frac{2}{x-1} + \frac{x+1}{(x^2+c)}\]

\[\text{WHO CARES?}\]

Remark: "We" know how to integrate

\[\int \frac{1}{x-a} \, dx \quad a \in \mathbb{R} \rightarrow \ln |x-a|\]
\[\int \frac{1}{x^2+b^2} \, dx \quad b \in \mathbb{R} \rightarrow \frac{1}{b} \tan^{-1} \left( \frac{x}{b} \right)\]
\[\int \frac{x}{x^2+a^2} \, dx \quad c \in \mathbb{R} \rightarrow \frac{1}{2} \ln(x^2+a^2)\]
HW 6 due Fri Apr 22
Exam 3 Fri Apr 29.

Today: HW 5 Solutions

Then: Compute \( \int \frac{3x^2 + 7}{(x-1)(x^2+4)} \, dx \)

\((x-1)\) and \((x^2+4)\) are coprime.

\[
\frac{x+1}{x-1, \sqrt{x^2+4}} = \frac{x^2-x}{x^2+4} = \frac{x+4}{5}.
\]

\[
x^2+4 = (x+1)(x-1) + 5.
\]

\[
\Rightarrow \quad 1 = -(x+1)(x-1) + \frac{x^2+4}{5}
\]

\[
\frac{1}{(x-1)(x^2+4)} = -\frac{x+1}{5} + \frac{1}{5} \frac{1}{x-1}
\]

\[
\frac{3x^2 + 7}{(x-0)(x^2+4)} = -\frac{(x+1)(3x^2+7)}{5}
\]
HW 6 due Fri Apr 22
Exam 3 Fri Apr 29.

Today: HW 5 solutions

Then: Compute \( \int \frac{1}{x^2 - 7(x^2 + 4)} \, dx \).

\((x-1) \& (x^2 + 4) \) are coprime.

\[
\begin{align*}
\frac{x + 1}{x - 1} & \cdot \frac{x^2 + 4}{x^2 - x} \\
& \frac{x - 1}{x + 1} \\
& \frac{x - 1}{5}.
\end{align*}
\]

\( x^2 + 4 = (x + 1)(x - 1) + 5 \).

\[
\begin{align*}
5 &= 1 \cdot (x^2 + 4) - (x + 1)(x - 1)
\end{align*}
\]

Coprime \( 1 = \frac{1}{5} (x^2 + 4) - \frac{x + 1}{5} (x - 1) \).

Divide by \((x - 1)(x^2 + 4) \) to get

\[
\frac{1}{(x-1)(x^2+4)} = \frac{1}{5} \left( \frac{1}{x-1} - \frac{x+1}{5(x^2+4)} \right).
\]

partial fractions
Now we can integrate.

Note.

\[ \int \frac{1}{x-1} \, dx = \ln |x-1| \]

\[ \int \frac{1}{x^2+4} \, dx \]

Let \( x = 2u \), so \( dx = 2 \, du \).

\[ = \int \frac{2 \, du}{4u^2+4} \]

\[ = \frac{1}{2} \int \frac{1}{u^2+1} \, du \]

\[ = \frac{1}{2} \arctan (u) \]

\[ = \frac{1}{2} \arctan \left( \frac{x}{2} \right) \]

\[ \int \frac{x}{x^2+4} \, dx \]

Let \( u = x^2+4 \), so \( du = 2x \, dx \).

\[ = \frac{1}{2} \int \frac{1}{u} \, du = \frac{1}{2} \ln |u| \]

\[ = \frac{1}{2} \ln (x^2+4) \]
Finally

\[ \int \frac{1}{(x-1)(x^2+4)} \, dx = \frac{1}{5} \int \frac{1}{x-1} \, dx - \frac{1}{5} \int \frac{1}{x^2+4} \, dx + \frac{1}{5} \int \frac{x}{x^2+4} \, dx \]

\[ = \frac{1}{5} \left( \ln |x-1| + \frac{1}{2} \arctan \left( \frac{x}{2} \right) + \frac{1}{2} \ln (x^2+4) \right) \]
HW 6 due Fri Apr 22
Exam 3 Fri Apr 29

Today: IRFA

Theorem (Euler, etc.

Every $f(x) \in \mathbb{R}[x]$ can be factored into deg 1 and deg 2 polynomials.

Equivalently, if $f(x) \in \mathbb{R}[x]$ is irreducible ("prime") over $\mathbb{R}$ then $\deg(f) = 1$ or 2.

Why did he care?

He wanted to integrate $
\int \frac{p(x)}{q(x)} \, dx \text{ where } p(x), q(x) \in \mathbb{R}[x].$

If $q(x)$ has deg 1 & 2 factors, this can be done through partial fractions.

eg. Last time we computed
\[ \int \frac{1}{(x^2+1)(x^2+4)} \, dx \]

\[ = \frac{1}{5} \left[ \ln |x-1| + \frac{1}{2} \arctan \left( \frac{x}{2} \right) + \frac{1}{2} \ln(x^2+4) \right] \]

Hence

\[ \text{IRFA} \implies \text{all } \frac{p(x)}{q(x)} \in \text{IR}(x) \implies \text{can be integrated.} \]
(in terms of ln, arctan, etc...)

How did he do it?
Let's see...

Start with \( \deg(f) = 1 \)

If \( f(x) = ax + b \), \( a, b \in \text{IR} \), \( \text{Done.} \)

If \( f(x) = ax^2 + bx + c \), \( a, b, c \in \text{IR} \), \( \text{Done.} \)

Let \( f(x) \in \text{IR}[x] \) and try to factor.

If \( \deg(f) = 1 \) or \( 2 \), nothing to do.

So \( \text{sp., } \deg(f) = 3 \).
Graph of $f$ looks like.

In either case IVT $\Rightarrow$ $f(\alpha) = 0$. Then use Factor Theorem to get

$$f(x) = (x-\alpha)g(x), \quad g \in \mathbb{R}[x]\quad \text{deg}(g) = 2 \checkmark$$

Next case: $\deg(f) = 4$. Looks like.

Maybe it has no real roots.
Eq. \( x^4 + 4 \).

But it still factors
\[
(x^4 + 4) = (x^2 - 2x + 2)(x^2 + 2x + 2)
\]

Sometimes it's much harder.

Eq. \( f(x) = x^4 - 4x^3 + 2x^2 + 4x + 4 \).

\[
= (x^2 - (2 + \sqrt{257}i)x + \cdots)^2
\]

Bernoulli: — Euler

No

Yes!

Is there a general method?

Theorem (Euler)
Every \( f(x) \in \mathbb{R}[x] \), \( \deg(f) = 4 \)
factors into 2 real quadratics.

Proof...

We may assume
\[
f(x) = x^4 + Bx^2 + Cx + D
\]

(Why?)
Now we are looking for Real $u, \alpha, \beta$. Where

\[ x^4 + Bx^2 + Cx + D = (x^2 + u\alpha + \alpha)(x^2 - u\alpha + \beta) \]

\[ = x^4 + (u - u) x^3 + (\alpha - u^2 + \beta) x^2 + (u\beta - u\alpha) x + u\beta. \]

Equate Coeffs:

\[ B = \alpha + \beta - u^2, \quad C = u(\beta - \alpha), \quad D = u\beta. \]

Hence:

1. \[ \alpha + \beta = B + u^2 \]
2. \[ \beta - \alpha = C/u. \]

1) + 2) \[ 2\beta = u^2 + B + \frac{C}{u} \]

1) - 2) \[ 2\alpha = u^2 + B - \frac{C}{u} \]

Then \[ D = \alpha \beta \Rightarrow \]

\[ 4D = 2\alpha 2\beta = \left( u^2 + B + \frac{C}{u} \right) \left( u^2 + B - \frac{C}{u} \right) \]

\[ = u^4 + 2Bu^2 + B^2 - \frac{C^2}{u^2}. \]
\[ u^6 + 2Bu^4 + (B^2 - 4D)u^2 - C^2 = 0 \]

Even degree, leading coeff. \( \frac{1}{2} > 0 \), const. term \( -C^2 < 0 \).

\[ \Rightarrow \text{ a real solution } u \in \mathbb{R}. \]

Then
\[ \alpha = \frac{1}{2} \left( u^2 + B + \frac{C}{u} \right) \]
\[ \beta = \frac{1}{2} \left( u^2 + B - \frac{C}{u} \right) \]

Also exist and are Real. We have solved the problem.

Every real quartic factors 😊.
HW 6 due next Fri
Exam 8 in 2 Fridays.

Today: RFTA.

In 1749, Euler claimed:

Theorem 7 (pg. 26) in his paper:

Every \( f(x) \in \text{R}[x] \) of degree \( 2^n \) factors as \( f(x) = g(x) h(x) \) where:

1. \( g(x), h(x) \in \text{R}[x] \)
2. \( \deg(g) = \deg(h) = 2^{n-1} \).

Claim: This result \( \Rightarrow \) RFTA.

Proof: Consider \( f(x) \in \text{R}[x] \) of degree \( m \), and choose \( n \) such that

\[
2^{n-1} < m \leq 2^n.
\]

\[
n-1 < \log_2(m) \leq n.
\]

Then let \( g(x) = x^{2^{n-m}} f(x) \).

Note: \( \deg(g) = 2^n - m + m = 2^n. \)
By Euler, \( g(x) = \Pi \text{ real quadratics} \)

Prime factors of \( g \) have deg 1 & 2.

But every prime factor of \( f \) is a prime factor of \( g \).

\( \Rightarrow \) prime factors of \( f \) have deg 1 & 2.

How did Euler prove Thm 7? NICE

Last time saw his proof

\[
\begin{align*}
f(x) &= g(x) \cdot h(x) \\
\text{deg } 4 &\quad \text{deg } 2 \\
\text{deg } 2 &\quad \text{deg } 2.
\end{align*}
\]

TRY for \( \text{deg } 8 = 2^3 \).

Assume \( f(x) \in \mathbb{R}[x] \) has form

\[
x^8 + Bx^6 + Cx^5 + Dx^4 + Ex^3 + Fx^2 + Gx + H.
\]

Suppose it factors as

\[
(x^4 - ux^3 + ax^2 + bx + c) \\
\times (x^4 + ux^3 + sx^2 + tx + y).
\]
Goal: Solve for \( u, \alpha, \beta, \delta, E, \xi, \gamma \in \mathbb{R} \).

Expand and equate coefficients:

\[
\begin{align*}
B &= \alpha + \delta - u^2 \\
C &= u(\alpha - \delta) + (\beta + \xi) \\
D &= u(\beta - 2\xi) + (\delta + \eta) + \alpha \delta \\
E &= u(\beta - \eta) + \beta \delta + \alpha E \\
F &= \xi + \beta \xi + \gamma \delta \\
G &= \beta \xi + \delta \xi \\
H &= \gamma \xi
\end{align*}
\]

Eliminate \( \alpha \) & \( \delta \)

then \( B \& E \)

then \( F \& G \)

Get an equation for \( U \) in terms of \( B, C, D, E, F, G, H \).

Hope it has a real root . . . .

-- Bernoulli

"But who among mortals wants to resolve equations in this manner? To believe that speculation on this is more curious than useful . . . ."

29 Nov 1743
So Euler had a new idea.

Back to deg 4.

Consider \( x^4 + B x^2 + C x + D \in \mathbb{R}[x] \).

Suppose \( f(x) \) has roots \( a, b, c, d \) somewhere.

\[ a, b, c, d \in \mathbb{F}. \]

Then \( f(x) = (x-a)(x-b)(x-c)(x-d) \).

\[ = x^4 - (a+b+c+d)x^3 + (ab+ac+ad+bc+bd+cd)x^2 - (abc+abd+acd+bcd)x + abcd. \]

If \( f(x) = (x^2-ux+ax)(x^2+ux+b) \)

Then must have

\[ u \in \mathbb{F}, \quad a+c \leq a+d, \quad b+c, b+d, c+d \geq. \]
Let
\[ p = a + b, \quad -p = c + d, \]
\[ q = a + c, \quad -q = b + d, \]
\[ r = a + d, \quad -r = b + c. \]

Then \( u \) satisfies
\[ (u - p)(u + p)(u - q)(u + q)(u - r)(u + r) = 0. \]
\[ (u^2 - p^2)(u^2 - q^2)(u^2 - r^2) = 0. \]

\( u_6 \) + \[ \begin{array}{c}
\text{even degree} \\
\text{neg. const. term}
\end{array} \] \( -p^2 q^2 r^2 \leq 0 \)

\[ \implies u \in \mathbb{R} \text{ exists.} \]

Issue: Why does \( X \) have \( IR \)-coeffs?
HW 6 due Fri
Exam 3 next Fri

Now: Symmetric Functions

Suppose that

\[ f(x) = x^4 + Ax^3 + Bx^2 + Cx + D \in \mathbb{F}[x] \]

has roots \( a, b, c, d \in F \subseteq \mathbb{R} \). In some field. Maybe we don’t know anything about \( F \).

Factor Theorem \( \Rightarrow \)

\[ f(x) = (x-a)(x-b)(x-c)(x-d) \cdot \]

\[ = x^4 - (a+b+c+d)x^3 + (ab+ac+ad+bc+bd+cd)x^2 - (abc+abd+acd+bcd)x + abc \]

Compare coefficients.
\[-A = a+b+c+d = e_1(a,b,c,d)\]
\[B = ab+ac+ad+bc+bd+cd = e_2(a,b,c,d)\]
\[-C = abc+abd+acd+bcd = e_3(a,b,c,d)\]
\[D = abcd = e_4(a,b,c,d)\]

Elementary Symmetric Polynomials.

Sometimes we write

\[e_i \text{ for } e_i(a,b,c,d)\]

if variables are understood.

Remark: even though \(a, b, c, d\) are mysterious, we know \(a+b+c+d = -A \in \mathbb{R}\).

General Definition:

Given \(r_1, r_2, \ldots, r_n \in \mathbb{F}\), we say

\[f(x) = (x-r_1)(x-r_2)\cdots(x-r_n)\]

\[= x^n - e_1 x^{n-1} + e_2 x^{n-2} - e_3 x^{n-3} + \cdots\]

\[\cdots + (-1)^{n-1} e_{n-1} x + (-1)^n e_n x^0.\]
$e_k = e_k(r_1, r_2, \ldots, r_n) = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} r_{i_1}r_{i_2}r_{i_3} \ldots r_{i_k}$

sum the roots taken $k$ at a time,

$= \text{the } k^{\text{th}} \text{ elementary symmetric poly.}$

Important Fact (Newton $\rightarrow$ Waring $\rightarrow$ Gauss) "Fundamental Theorem of Symmetric Polynomials/Functions".

Let $F(r_1, r_2, \ldots, r_n)$ be any function symmetric in $r_1, r_2, \ldots, r_n$ (i.e. $f(r_{i_1}, r_{i_2}, \ldots, r_{i_k}) = f(r_{\pi(1)}, r_{\pi(2)}, \ldots, r_{\pi(k)})$ for any $1 \leq i \leq n-1$).

Then $F$ has a unique expression in terms of $e_1, e_2, \ldots, e_n$. \(\square\)
Eg. Suppose \( x^3 + 23x + 1 \) has roots \( r, s, t \). What polynomial has roots \( 1+r, 1+s, 1+t \)?

First note:

\[
\begin{align*}
e_1 &= r + s + t = -0 \\
e_2 &= rs + rt + st = +23 \\
e_3 &= rst = -1
\end{align*}
\]

We are looking for

\[
g(x) = (x - (1+r))(x - (1+s))(x - (1+t)) = x^3 - (1+r+1+s+1+t)x^2 + ((1+r)(1+s) + (1+r)(1+t) + (1+s)(1+t))x - (1+r)(1+s)(1+t).
\]

Symmetric in \( r, s, t \).

\( \Rightarrow \) express in terms of \( e_1, e_2, e_3 \)

\[
1+r+1+s+1+t = 3 + r + s + t = 3 + e_1 = 3 - 0 = 3
\]
\[(1+r)(1+s)(1+t) = \]
\[= 1 + rs + rs + 1 + rt + st + 1 + rs + rt + st \]
\[= 3 + 2(r + s + t) + (rs + rt + st) \]
\[= 3 + 2e_1 + 1e_2 \]
\[= 3 + 2 \cdot 0 + 1 \cdot 23 = 26. \]

\[(1+r)(1+s)(1+t) \]
\[= 1 + (r + s + t) + (rs + rt + st) + (rs \cdot t) \]
\[= 1 + e_1 + e_2 + e_3 \]
\[= 23 \cdot 0 + 23 - 1 = 23. \]

Hence

\[g(x) = x^3 - 8x^2 + 26x - 23 \]

has roots 1+r, 1+s, 1+t

Exercise:

Express \(r^3 + s^3 + t^3\) in terms of \(e_1, e_2, e_3\).
Gauss' Proof of FTSP

Let \( f(x_1, x_2, \ldots, x_n) \) be symmetric poly.
in \( n \) variables \( x_1, x_2, \ldots, x_n \).

A general term looks like

\[
f(x) = \cdots + a_{i_1, i_2, \ldots, i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} + \cdots
\]

\( (i_1, i_2, \ldots, i_n) \)
called the "degree sequence".

Order degree sequences "lexicographically" (like a dictionary).

e.g. \((0, 0, 0) < (0, 0, 1) < (0, 1, 1) < (0, 1, 2)\)

Def: The leading term of \( f(x) \) has "biggest" degree sequence, say

\[
a x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}
\]

Claim: \( i_1 \geq i_2 \geq \cdots \geq i_n \)

Proof: Suppose \( i_k < i_k \) for some \( k \).

Then by symmetry, \( a x_1^{i_1} \cdots x_k^{i_k} x_{k+1} x_{k+2} \cdots x_n \neq a x_1^{i_1} \cdots x_k^{i_k} x_{k+1} x_{k+2} \cdots x_n \)
Is also a term of F with a "differ" degree sequence.

Hence leading term has $i_1, i_2, \ldots, i_n$ in.

Note: $a e_1 + e_2 + \ldots + e_{n-1} + e_n$
also has leading term

$a x_1 x_2 \ldots x_n$.

Hence $f - ae_1 + e_2 + \ldots + e_n$
is symmetric with smaller leading term.

Proved by induction.
HW 6 due this Fri
Exam 3 next Fri
O.H
Today & Tm. 2:30 – 4:00.

Now: FTSP
Fund. Thm. of Symm. Polynomials.

Recall: if

\[
(x-r_1)(x-r_2) \ldots (x-r_n) = x^n - e_1 x^{n-1} + e_2 x^{n-2} - \ldots + (-1)^n e_n x^0.
\]

Then \( e_1, e_2, \ldots, e_n \) are polynomials
in the \( r_1, r_2, \ldots, r_n \).

\[
e_1 = r_1 + r_2 + \ldots + r_n
\]
\[
e_2 = r_1 r_2 + r_1 r_3 + \ldots + r_1 r_n + r_2 r_3 + \ldots + r_{n-1} r_n
\]
\[
e_n = r_1 r_2 \ldots r_n.
\]

Elementary symmetric polynomials
Theorem (FTSP).

Any symmetric poly $f(r_1,r_2,\ldots,r_n)$ can be expressed as

$$f = g(e_1,e_2,\ldots,e_n)$$

for some unique poly $g$.

Example: let $(r_1,r_2) = (a,b)$

Then $f(a,b) = a^3 + b^3$ is symmetric.

Note: $e_1 = a + b$

$e_2 = ab$.

$$e_1^3 = (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$  

$$f - e_1^3 = -3a^2b - 3ab^2$$

$$= -3ab(a+b) = -3e_2e_1.$$  

Hence $f - e_1^3 = -3e_1e_2$.

$$f = e_1^3 - 3e_1e_2$$

$$= g(e_1,e_2).$$  

unique
Another eg.

Consider \( x^3 + Ax^2 + Bx + C \)
with roots \( r, s, t \).

So 
\[-A = e_1(r, s, t) = r + s + t \]
\[B = e_2(r, s, t) = rs + rt + st \]
\[-C = e_3(r, s, t) = rst \].

Problem: Express \( \Delta = (r-s)^2(r-t)^2(s-t)^2 \)
in terms of \( A, B, C \).

Where to begin?

Expand \( \Delta \) and look for the
"biggest" term

what does this mean?

General term is: \( \lambda_1 x^m y^n \).

"degree" = \( (l, m, n) \).

How to say which degrees are big?

\((2, 1, 0) \leq (0, 1, 2)\)?
Use lexicographic (dictionary) order

\[ (0, 0, 0) < (0, 0, 1) < (0, 1, 0) < (0, 1, 2). \]

For us...
\[ (4, 2, 0). \]

\[ \Delta = r^4 s^2 + \text{lower terms}. \]

leading term.

\[ e_1^2 e_2^2 = (r + s + t)^2 (rs + rt + st)^2 \]

\[ = r^4 s^2 + \text{lower terms}. \]

same leading term.

Cancel leading term:
\[ (4, 1, 1) \]

\[ \Delta - e_1^2 e_2^2 = -4 r^4 s t + \text{lower terms} \]

\[ (4, 2, 0) > (4, 1, 1) \]

leading term went down.

\[ -4 e_1^2 e_2^2 = -4 (s + s + t)^3 r s t \]

\[ = -4 r^3 s t + \text{lower terms}. \]

\[ = -4 r^4 s t + \text{lit.} \]
\[ \Delta = 4 e_2^2 + 4 e_3^2 e_2 - 4 e_3^3 + 18 e_1 e_2 e_3 - 27 e_3^2 \]

\[ \Delta = (A^2 B^2 - 4 A^3 C - 4 B^3 + 18 A B C) - 27 C^2 \]

The discriminant of a cubic poly. 😊
So... how to prove \( \text{T-FSP} \)?

**Proof (Gauss): Algorithmic**

Induction on degree sequence. Let \( f(r_1, \ldots, r_n) \) be symmetric with lex-leading term

\[ C_{r_1} r_2 \cdots r_n \]

Claim: \( i_1 \geq i_2 \geq \cdots \geq i_n \).

Otherwise we would have \( i_k < i_{k+1} \) for some \( k \). By symmetry \( f \not\in D_{i_k} \).

\[ C_{r_1} \cdots r_{i_k} \cdots r_{i_k+1} \cdots r_n \]

is also a term of \( f \) with "bigger" deg. sequence, **contradiction**.

\[ e_i e_i e \cdots e \] has same leading term.

\[ f - e_i e_i e \cdots e \] is symmetric with smaller leading term. By induction we're done.
HW6 due now.
Exam 3 next Fri.

Today: *(Grand Finale)*

We know how to go from

the roots of a polynomial

\[ r_1, r_2, \ldots, r_n \]

\[ -e_1, -e_2, -e_3, \ldots, (-1)^{n-1}e_n. \]

elem. symm. polys.

But can we go back? 

the coefficients \[ e_1, e_2, \ldots, e_n \]

\[ r_1, r_2, \ldots, r_n \]

eg. Given \[ e_1 = r_1 + r_2 \]

Find \[ r_1 \text{ and } r_2. \]

\[ e_2 = r_1 r_2 \]

Now let \[ s_1 = r_1 + r_2 \] 

\[ s_2 = r_1 - r_2 \]

Method due to Lagrange.

\[
\begin{pmatrix}
  s_1 \\
  s_2
\end{pmatrix} =
\begin{pmatrix}
  1 & 1 \\
  1 & -1
\end{pmatrix}
\begin{pmatrix}
  r_1 \\
  r_2
\end{pmatrix}
\]
Invert: \[
\begin{pmatrix}
  r_1 \\
  r_2 
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
  -1 & -1 \\
  -1 & 1 
\end{pmatrix}
\begin{pmatrix}
  s_1 \\
  s_2 
\end{pmatrix}
\]

\[
\Rightarrow r_1 = \frac{1}{2} (s_1 + s_2)
\]

\[
r_2 = \frac{1}{2} (s_1 - s_2)
\]

But what are \( s_1, s_2 \)?

\[
s_1 = r_1 + r_2 = e_1 \quad \checkmark
\]

\[
s_2 = r_1 - r_2 \quad \text{NOT Symmetric}
\]

\[
s_2^2 = (r_1 - r_2)^2 \quad \text{Symmetric} \quad \checkmark
\]

\[
= r_1^2 - 2 r_1 r_2 + r_2^2
\]

\[
= (r_1 + r_2)^2 - 4 r_1 r_2
\]

\[
= e_1^2 - 4 e_2
\]

Let \( s_2 = \sqrt{e_1^2 - 4 e_2} \).

Choose some value.

Then

\[
r_1 = \frac{1}{2} (s_1 + s_2) = \frac{1}{2} (e_1 + \sqrt{e_1^2 - 4 e_2}) \quad \text{Quad. Formula}
\]

\[
r_2 = \frac{1}{2} (s_1 - s_2) = \frac{1}{2} (e_1 - \sqrt{e_1^2 - 4 e_2})
\]
Apply Lagrange to the cubic.

Let \( r_1, r_2, r_3 \) be the roots of the cubic.

Express in terms of \( e_1, e_2, e_3 \), the coefficients.

Set \( \omega = e^{2\pi i/3} \) and let

\[
\begin{align*}
S_1 &= r_1 + r_2 + r_3 \\
S_2 &= r_1 + \omega r_2 + \omega^2 r_3 \\
S_3 &= r_1 + \omega^2 r_2 + \omega r_3.
\end{align*}
\]

Invert:

\[
\begin{align*}
r_1 &= \frac{1}{3} (S_1 + S_2 + S_3) \\
r_2 &= \frac{1}{3} (S_1 + \omega S_2 + \omega^2 S_3) \\
r_3 &= \frac{1}{3} (S_1 + \omega^2 S_2 + \omega S_3)
\end{align*}
\]

Now solve for \( S_1, S_2, S_3 \).

\( S_1 = e_1 \), \( \checkmark \)

\( S_2, S_3 \) are NOT symmetric in \( r_1, r_2, r_3 \)

But \( A = S_2^3 + S_3^3 \) AND \( B = S_2^3 + S_3^3 \) ARE symmetric.

Use Gauss' Algorithm to get,
\[ A = 2e_1^3 - 9e_1e_2 + 27e_3, \]
\[ B = e_1^2 - 3e_2. \]

Then \( s_2^3 \) & \( s_3^3 \) are the roots of

\[
(x - s_2^3)(x - s_3^3) = x^2 - (s_2^3 + s_3^3)x + s_2^3s_3^3 = x^2 - A x + B
\]

So, \( s_2^3, s_3^3 = \frac{1}{2}(A \pm \sqrt{A^2 - 4B}) \). Quadratic Formula

\[ s_2, s_3 = \frac{1}{2}(A + \sqrt{A^2 - 4B}) \]

Recall \( s_1 = e_1 \).

Finally, Cardano's Formula

\[ r_1 = \frac{1}{3} \left( s_1 + s_2 + s_3 \right) \text{ in terms of } e_1, e_2, e_3
\]

\[ r_2 = \frac{1}{3} \left( s_1 + \omega s_2 + \omega^2 s_3 \right) \]

\[ r_3 = \frac{1}{3} \left( s_1 + \omega^2 s_2 + \omega s_3 \right) \text{ We "solved the cubic"} \]
Lagrange also works on the quartic but it's complicated. (See Chapter 6.5)

Lagrange tried the quintic and failed.

Then he tried to show quintic cannot be solved.

i.e. Conjecture: There does NOT exist a formula for \( r_1, r_2, r_3, r_4, r_5 \) in terms of \( e_1, e_2, e_3, e_4, e_5 \)

\[
\begin{align*}
e_1 &= r_1 + r_2 + \ldots + r_5 \\
e_2 &= r_1 r_2 + r_1 r_3 + \ldots + r_4 r_5 \\
e_3 &= r_1 r_2 r_3 + r_1 r_2 r_4 + \ldots + r_4 r_5 r_5 \\
e_4 &= r_1 r_2 r_3 r_4 + r_1 r_2 r_3 r_5 + \ldots + r_4 r_5 r_6 r_7 \\
e_5 &= r_1 r_2 r_3 r_4 r_5,
\end{align*}
\]

and using only \( +, -, \times, \frac{1}{2}, \sqrt[3]{}, \sqrt{5}, 4\sqrt{5}, \text{ etc.} \)

an expression in Radicals

Paraphrase:
The general quintic is (unsolvable)
What happened next?

Lagrange
\begin{itemize}
  \item Ruffini
\end{itemize}
\begin{itemize}
  \item Niels Henrik Abel (1802-1829):
\end{itemize}
\begin{itemize}
  \item Theorem (Abel): The general quintic is unsolvable.
\end{itemize}

Issues:

- His proof was complicated.
- It had a gap.
- Some quintics are solvable.

\[ x^5 - 1 \text{ has roots: } 1, \omega, \omega^2, \omega^3, \omega^4 \text{ where } \omega^5 = 1 \]

Which ones?
Évariste Galois (1811 – 1832)

Let $f(x) \in F[x]$ some field.

Let $E = F$ be the splitting field for $f(x)$. (It contains all the roots of $f$.)

Let $\text{Gal}(E/F) = \text{automorphisms of } E \text{ that leave } F \text{ fixed}.$

$\text{Gal}(E/F)$ is called the Galois group of $f(x)$.

Theorem (Galois).

$f(x)$ is solvable $\iff \text{Gal}(E/F)$ is "solvable".

I won't define this.

ICOSAHEDRON