HW 3 due March 4
(not posted yet)
Exam 2 March 25.

Today: Greek

The Pythagoreans (~500)
- "all is number"
- "number" = positive integers and their ratios

The Crisis: \(\sqrt{2}\) is not a "number".

Proof: Suppose \(\sqrt{2} = \frac{a}{b}\) for \(a, b \in \mathbb{Z}\).
Square to get \(2 = \frac{a^2}{b^2}\), or \(2b^2 = a^2\).
Since \(a^2\) is even, \(a\) is even, say \(a = 2a'\).
Then \(2b^2 = (2a')^2 = 4(a')^2\), or \(b^2 = 2(a')^2\).
Since \(b^2\) is even, \(b\) is even, say \(b = 2b'\).

We get \(\sqrt{2} = \frac{a}{b} = \frac{2a'}{2b'} = \frac{a'}{b'}\) with \(a' > a' > 1\)
\(b > b' > b' > b'' > \cdots > 1\).

Repeat to get \(\sqrt{2} = \frac{a}{b} = \frac{a'}{b'} = \frac{a''}{b''} = \cdots = e + c\).

But this is absurd.
(reductio ad absurdum) \(\square\)
So $\sqrt{2}$ is not a "number". But it's a perfectly good "length".

Greek is replaced:

"number" — "length of line segment".
This persisted until modern times.

Greek math based on... Ruler & Compass.
(i.e. lines and circles).

Start from

unit length: "1".

Which lengths can we construct?
(i.e. which "numbers" "exist"?)

1, 2, 3, 4, ... all positive $\mathbb{Z}$.
Given $\alpha$, $\beta$ we can add: form radius $\beta$
circle of $A$.

$\alpha + \beta$.

$O \rightarrow A \rightarrow B$.

$O \rightarrow B = \alpha + \beta$.

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Given $\alpha - \beta$ we can subtract: same idea.

$\alpha - \beta$.

$O \rightarrow \text{radius } \beta$.

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Given $\alpha \cdot \beta$ we can multiply:

1. form perpendicular axes.
② we can draw parallel to a given line

**Proof (Euclid I.31).**

Given

③ Given:

\[ \alpha \beta \]

\[ \frac{oD}{oB} = \frac{oE}{oC} \implies \frac{\alpha}{\beta} = \frac{OE}{oC} \implies OE = \alpha \beta. \]

Draw DE parallel to BC.

Given \( \alpha, \beta \) we can divide. Same idea.

Given:

Construct RC parallel to DE.

Then:

\[ \frac{oD - oE}{oB} = \frac{oC}{oC} \]

\[ \frac{\beta}{\beta} = \frac{\alpha}{\alpha} \]

Get \( oC = \frac{\alpha}{\beta} \).
Conclusion: all positive rationals \( \mathbb{Q}^+ \) are constructible.

Is that all? No!

Given \( \alpha, \beta \), we can form \( \sqrt{\alpha^2 + \beta^2} \):

\[
\sqrt{\alpha^2 + \beta^2}, \quad \text{by Pythagorean Theorem.}
\]

\( \alpha \)

\( \beta \)

\( \sqrt{\alpha^2 + \beta^2} \)

eg. \( \sqrt{2} = \sqrt{1^2 + 1^2} \) is constructible?

Is \( \sqrt{3} \) constructible?

\( 3 \neq \alpha^2 + \beta^2 \), \( \rightarrow \) no help.

Theorem: If \( \alpha \) is constructible, then \( 5 \) \( \sqrt{\alpha} \) is constructible.

Proof: \( \Box \) we can bisect a segment

(Ferulid, I.10)
Given

Draw $DC \perp AB$.

Exercise: Show $ACB = 90^\circ$.

Hence, $\triangle ABC$, $\triangle ADC$, $\triangle CDB$ are similar.

We get

$$\frac{AP}{CD} = \frac{x}{1} = \frac{CD}{BD}.$$
Consider vectors \( \vec{u} = (\cos \theta + 1, \sin \theta) \)
\( \vec{v} = (\cos \theta - 1, \sin \theta) \).

What is the angle between \( \vec{u} \), \( \vec{v} \)?

Recall:
\[ \vec{u} \cdot \vec{v} = ||\vec{u}|| ||\vec{v}|| \cos \alpha. \]

\[ \vec{u} \cdot \vec{v} = 0 \implies \vec{u} \perp \vec{v}. \]

\[ \vec{u} \cdot \vec{v} = (\cos \theta - 1)(\cos \theta + 1) + \sin \theta \sin \theta. \]

\[ = \cos^2 \theta - 1^2 + \sin^2 \theta \]
\[ = \cos^2 \theta + \sin^2 \theta - 1 = 0. \]
HW 2 due next Fri Mar 4

Today: Constructibility

Using a straightedge & compass

which "lengths" = "numbers" are constructible?

Start with an arbitrary unit length "1"

Last time we proved:
If $\alpha, \beta$ are constructible, then so are

1. $\alpha + \beta$
2. $\alpha - \beta$ (when $\beta < \alpha$)
3. $\alpha \cdot \beta$
4. $\alpha / \beta$

- All rational numbers are constructible.
5. $\sqrt{\alpha}$
So e.g.

\[
\sqrt{1 + \sqrt{3}} + \frac{101}{5 + \sqrt{2}} = \frac{77}{77}
\]

is constructible.

Q: Is every \( \alpha \in \mathbb{R}, \alpha > 0 \) constructible?

i.e. are "constructible lengths" = "all possible lengths"?

The Greeks didn't know, but they got stuck on 3 problems, ...

1) Squaring the circle.

Given a circle, construct a square with the same area.

Unit circle has area \( \pi \).

Is \( \sqrt{\pi} \) (or \( \pi \)) constructible?

Theorem (Lindemann, 1882): NO.
2) Doubling the Cube.

Give (the edge of) a cube, construct (the edge of) a cube with double the volume.

Volume 1

\[ \sqrt[3]{2} \]

Volume 2

Is \( \sqrt[3]{2} \) constructible?

Theorem (Descartes, 1637): \( \text{N O} \).

3) Trisecting an Angle

Given the angle \( \theta \), construct the angle \( \frac{\theta}{3} \),

given lines construct lines

\[ \frac{\theta}{3} \]

\[ \frac{\theta}{3} \]

Given \( \cos \theta \), is \( \cos \left( \frac{\theta}{3} \right) \) always constructible?

Theorem (Gauss, 1796 \& Wantzel, 1837): \( \text{N O} \).

Q: How'd they do dat?
A: Algebra

Let \( C = \text{constructible numbers} \)
\( D = \text{numbers formed from} \)
\[ 1, 1, -1, x, \frac{c}{c}, \sqrt{-} \]

Theorem: \( C = D \)

Proof: We already saw \( D \subseteq C \).
Need \( C \subseteq D \).

Think in coordinates (Descartes). Note:
point \((x, y)\) constructible \(\Rightarrow x, y\) are able.

So sp. \((x, y)\) has been constructed.

Where did \((x, y)\) come from? It was an intersection point for some
line & line
line & circle
circle & circle

with able coefficients. Claim: Then
\[ x = \frac{a + \sqrt{b}}{c} \quad \text{and} \quad y = \frac{c + \sqrt{d}}{e} \]
for some constructible $a, b, c, d, e, f$.
 Assume for induction that $a, b, c, d, e, f \in D$.
 Then $x, y \in D$.

Intersect
- line & line $\rightarrow$ linear equation
- line & circle $\rightarrow$ quadratic
- circle & circle $\rightarrow$ ?

You will do this on HW 3
Recall: If $x \in \mathbb{R}$ is constructible (with straight edge and compass) then $x$ is formed recursively by intersecting lines and circles with only coefficients. In each case the solution can be done using $+, -, \times, \div, \sqrt{}$

1. line \ line $\rightarrow$ linear equation
2. line \ circle $\rightarrow$ quadratic equation
3. circle \ circle $\rightarrow$ ? Exercise.

Hence $x$ has an expression in $1, +, -, \times, \div, \sqrt{}$

"a degree 2 algebraic expression"

Conversely, we have seen that $x + p, xp, \frac{x}{p}, \sqrt{x}$ can be constructed, so any $x \in \mathbb{R}$ with a degree 2 alg. expression is constructible.
Summary: Let \( C = \) constructible \#'s
\[ D = \text{\#'s formed recursively from } 1, +, -, \times, \div, \sqrt{\cdot} \]

Then \( C = D \) (geometry) (Algebra!)

(A means to show that some \( x \in \mathbb{R} \) is NOT c'de.

Describe \( D \) more precisely?

Let \( F = \) a field "number system with +, -, \times, \div"

(eg. \( \mathbb{R}, \mathbb{Q}, \mathbb{C} \)).

and suppose \( c \in F \) with \( \sqrt{c} \in F \).

We form a new number system

\[ F[\sqrt{c}] = \left\{ a + b\sqrt{c} : a, b \in F \right\} \]
\[
\mathbb{Q}[\sqrt{2}] = \{ a + b\sqrt{2} : a, b \in \mathbb{Q} \}
\]

"rational numbers adjoin \( \sqrt{2} \)"

eg. \( \mathbb{R}[\sqrt{-1}] = \{ a + b\sqrt{-1} : a, b \in \mathbb{R} \} \)

In general, \( \mathbb{F}[\sqrt{c}] \) is similar to \( \mathbb{R}[\sqrt{-1}] \).

We can divide:

eg. Given \( a + b\sqrt{2} \in \mathbb{Q}[\sqrt{2}] \)

\[
\frac{1}{a + b\sqrt{2}} = \frac{a - b\sqrt{2}}{(a + b\sqrt{2})(a - b\sqrt{2})} = \frac{a - b\sqrt{2}}{a^2 - 2b^2}.
\]

\[
= \left( \frac{a}{a^2 - 2b^2} \right) + \left( \frac{-b}{a^2 - 2b^2} \right) \sqrt{2} \in \mathbb{Q}[\sqrt{2}].
\]

\( \Rightarrow \mathbb{Q}[\sqrt{2}] \) is a field.

\[\mathbb{Q}[\sqrt{2}] \rightarrow \mathbb{Q}[\sqrt{2}]\]

The map \( a + b\sqrt{2} \mapsto a - b\sqrt{2} \)

is called \( \text{CONJUGATION} \).

\( \sqrt{2} \mapsto \sqrt{2} \)
$\mathbb{Q}[\sqrt{2}]$ is also a vector space.

Suppose $a + b\sqrt{2} = c + d\sqrt{2}$

Then $(a - c) = (d - b)\sqrt{2}$.

If $b \neq d$, then $\sqrt{2} = \frac{a - c}{d - b}$ a contradiction!

Hence $b = d$ & $a = c$

Summary:

$a + b\sqrt{2} = c + d\sqrt{2} \iff a = c & b = d$.

So $a + b\sqrt{2}$ acts like a vector $(a, b)$

$(\text{geometry})$

$\mathbb{Q}[\sqrt{2}] \cong \mathbb{Q}^2$

the rational plane

Summary: Given field $F$ and $c \in F$ with $\sqrt{c} \in F$, then $F[\sqrt{c}]$ is a field, with

$a + b\sqrt{c} = a' + b'\sqrt{c} \iff a = a' \text{ AND } b = b'$
\[ F \subseteq F[52] \]
is a "quadratic field extension."

So WHAT?

Rephrase constructibility:

\( \alpha \in \mathbb{Q} \) is constructible

\[ \Rightarrow \]

\( \exists \) chain of quadratic extensions

\( \mathbb{Q} = F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots \subseteq F_r \subseteq \ldots \subseteq \mathbb{R} \)

with \( \alpha \in F_r \).

This is USEFUL

Theorem: \( 3\sqrt{2} \) is not constructible.

(Landau, when he was a student).

Proof: Suppose (for contradiction) that \( 3\sqrt{2} \) is constructible. Then \( \exists \)

\[ \mathbb{Q} = F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots \subseteq F_r \subseteq F_{r+1} \subseteq \ldots \subseteq \mathbb{R} \]

where \( 3\sqrt{2} \in F_{r+1} \) but not in \( F_r \).
Hence \( \sqrt[3]{2} = a + b\sqrt{c} \) with \( a, b, c \in \mathbb{F}_{k+1} \) and \( c \neq 0 \).

**CUBE to get**

\[
2 + 3\sqrt{c} = (a + b\sqrt{c})^3 \]

\[
= (a^3 + 3a^2b\sqrt{c}) + (3ab^2c + b^3c)\sqrt{c}.
\]

**Compare coefficients:**

\[
2 = a^3 + 3ab^2c \quad \text{and} \quad 0 = 3a^2b + b^3c
\]

Now (for fun) expand

\[
(a - b\sqrt{c})^3 - 2
\]

\[
= (a^3 + 3ab^2c - 2) - (3a^2b + b^3c)\sqrt{c},
\]

from (*)

\[
= 0 - 0\sqrt{c} = 0.
\]

Hence, \( a - b\sqrt{c} \) is a Real cube root of 2.

Conclude \( \sqrt[3]{2} = a + b\sqrt{c} = a - b\sqrt{c} \).

\[ a = a \quad \text{and} \quad b = -b \Rightarrow b = 0
\]

\[ \Rightarrow \sqrt[3]{2} = a \in \mathbb{F}_{k+1} \text{ contradiction} \]
Suppose $3\sqrt{2}$ is constructible. Then $\exists$

$$Q = F_0 \subset F_1 \subset \cdots \subset F_k \subset F_{k+1} \subset \cdots \subset \mathbb{R}$$

where $3\sqrt{2} \in F_{k+1}$ but NOT $F_k$.

Then we can write $3\sqrt{2} = a + b\sqrt{c}$, $a, b \in F_k$.

So $$(a + b\sqrt{c})^3 - 2 = 0.$$ 

Apply conjugation $F_{k+1} \rightarrow F_k$

$$\alpha + \beta\sqrt{c} \mapsto \alpha - \beta\sqrt{c}$$

$$(a + b\sqrt{c})^3 - 2 = 0$$

$$(a - b\sqrt{c})^3 - 2 = 0$$

\implies a - b\sqrt{c} \in \mathbb{R} \text{ is a cube root of } 2.$$

\[\checkmark\]
Today : A "high-school" problem

Compute intersection of circles. (Easy?)

\[ \sum (x-a)^2 + (y-b)^2 = R^2 \]
\[ \sum (x-c)^2 + (y-d)^2 = r^2 . \]

Try to simplify by changing coordinates.

Idea: move centers \((a,b)\) and \((c,d)\)

to \((0,0)\) and \((0,0)\).

First let \(T: \mathbb{R}^2 \to \mathbb{R}^2 \) be translation by \((-a)\)

\[ T(x,y) = (x-a, y-b) \]
\[ T(b) = (b) \text{ and } T(c) = (c - a) \]

Let \( \Delta = \sqrt{(c-a)^2 + (d-b)^2} = \text{dist}\left(\left(\begin{array}{c} a \\ b \end{array}\right), \left(\begin{array}{c} d \\ b \end{array}\right)\right) \)

Next, rotate by \( -\theta \)

\[ R_{-\theta} : \mathbb{R}^2 \to \mathbb{R}^2. \]

\[ R_{-\theta} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \]

\[ = \frac{1}{\Delta} \begin{pmatrix} c-a & d-b \\ -(d-b) & c-a \end{pmatrix} \]

Does this work?

\[ R_{-\theta}(b) = (b) \text{ and } \]

\[ R_{-\theta}(c-a) = \frac{1}{\Delta} \begin{pmatrix} c-a & d-b \\ -(d-b) & c-a \end{pmatrix} \begin{pmatrix} c-a \\ d-b \end{pmatrix} \]
\[= \frac{1}{\Delta} \begin{pmatrix} \Delta^2 \\ 0 \end{pmatrix} = \begin{pmatrix} \Delta \\ 0 \end{pmatrix} \]

The transformation is

\[ \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow R(T( \begin{pmatrix} x \\ y \end{pmatrix} )) \]

\[= R \begin{pmatrix} x-a \\ y-b \end{pmatrix} \]

\[= \frac{1}{\Delta} \begin{pmatrix} (c-a) & d-b \\ -(d-b) & c-a \end{pmatrix} \begin{pmatrix} x-a \\ y-b \end{pmatrix} \]

\[= \frac{1}{\Delta} \begin{pmatrix} (x-a)(c-a) + (y-b)(d-b) \\ -(x-a)(d-b) + (y-b)(c-a) \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} \]

New coordinates.

\[\begin{pmatrix} x' \\ y' \end{pmatrix} = RT'( \begin{pmatrix} x \\ y \end{pmatrix} )\]

How to invert? \(\square\)
The inverse of \( RT \) is \( T^{-1} R^{-1} \)

where \( T^{-1} \) = translate by \((+a, +b)\)
\( R^{-1} \) = rotate by \(+\theta\).

*Formula on hadout.*

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**Note:** If \((x, y)\) is on circles

\((a, b)\) radius \( R \)
\((c, d)\) radius \( r \)

Then \((x', y') = RT(y)\) is on circles

\((0, 0)\) radius \( R \)
\((0, 0)\) radius \( r \)

Hence

\[ x'^2 + y'^2 = R^2 \]
\[ (x' - \Delta)^2 + y'^2 = r^2 \]

Much easier to solve \( \rightarrow \) A.6.
Recall:

Given field $F$ with $\alpha \in F$, $\sqrt{\alpha} \in F$, define quadratic field extension

$$F = F[\sqrt{\alpha}] = \{ a + b\sqrt{\alpha} : a, b \in F \}.$$

(similar to complex numbers.

$F = F[\sqrt{i}]$)

We say $\alpha \in F$ has a "degree 2 alg. expression" if $\alpha$ can be written from $1, i, -1, x, \sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{11}, \sqrt{13}$. 

e.g. $\alpha = \sqrt{1 + \sqrt{3}}$

Equivalently, $\exists$ chain of $\mathbb{Q}, F, E$.

$$\mathbb{Q} = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_i \subseteq F_{i+1} \subseteq \cdots \subseteq F$$

such that $\alpha \in F_i$ for some $i$.

e.g. chain

$$\mathbb{Q} \subseteq \mathbb{Q}[\sqrt{3}] \subseteq \mathbb{Q}[\sqrt{5}] \subseteq \mathbb{Q}[\sqrt{1 + \sqrt{3}}] \subseteq \mathbb{Q}[\sqrt{1 + \sqrt{5}}] \subseteq \mathbb{Q}$$

more "nesting".
Theorem: $3\sqrt{2}$ is not constructible.

Proof: suppose (for contradiction) that it is.

Then $\exists \quad \Omega = F_0 \subseteq F_1 \subseteq \cdots \subseteq \mathbb{R}$,

where $3\sqrt{2} \in F_{r+1}$ for some $r$.

$3\sqrt{2} \in F_r$.

(Note: $3\sqrt{2} \in F_0 = \Omega$).

Say, $F_{r+1} = F_r [\sqrt[n]{c}] = \langle a + b \sqrt[n]{c}, a, b \rangle$.

Hence $3\sqrt{2} = a + b \sqrt[n]{c}$ for some $a, b \in F_r$.

(Note: $b \neq 0$ because $3\sqrt{2} \in F_r$).

Then $(a + b \sqrt[n]{c})^3 = 2$.

Conjugate both sides:

\[ (a + b \sqrt[n]{c})^3 = 2 + 0 \sqrt[n]{c}. \]

\[ \left( \frac{a + b \sqrt[n]{c}}{2} \right)^3 = 2 - 0 \sqrt[n]{c}. \]

\[ (a - b \sqrt[n]{c})^3 = 2 \]

Hence $a - b \sqrt[n]{c} \in \mathbb{R}$ is a cube root of 2.

Hence $a - b \sqrt[n]{c} = a + b \sqrt[n]{c} \Rightarrow b = -b \Rightarrow b = 0$.
HW 3 due now.
HW 4 due next Friday, March 11.
Spring Break
Exam 2, Fri. Mar. 25

Today: $\cos \theta \rightarrow \cos \frac{\theta}{3}$ ?

We have seen

$$\cos \left( \frac{\theta}{2} \right) = \pm \sqrt{\frac{1 + \cos \theta}{2}}.$$ 

Hence if $\cos \theta$ is c'ble then $\cos \theta$ is c'ble.

"Any c'ble angle can be bisected."

Some angles can be trisected.

e.g. a right angle.

![Diagram](image-url)

These points are c'ble.
Proof: Let \( \omega = \cos \left( \frac{2\pi}{3} \right) + i \sin \left( \frac{2\pi}{3} \right) \)

Picture:

\[-1 = \omega^2 \]

\[\omega^3 = -\omega \]

Cube roots of \(-1\).

\[\omega + \omega^2 + \omega^3 = 0 \]
\[\omega - 1 + \frac{1}{\omega} = 0 \]
\[\omega + \frac{1}{\omega} = 1 \]
\[2 \cos \left( \frac{\pi}{3} \right) = 1 \]
\[\cos \left( \frac{\pi}{3} \right) = \frac{1}{2} \Rightarrow \sin \left( \frac{\pi}{3} \right) = \sqrt{1 - \frac{1}{2}} = \frac{\sqrt{3}}{2} \]

Both c'ble!

However, Claim: \( \frac{\pi}{3} \) cannot be trisected.

i.e. \( \cos \left( \frac{\pi}{9} \right) \) is not c'ble.

Let's prove it!
Recall: \[ \cos(3\theta) = 4\cos^3\theta - 3\cos\theta. \]

Put \( \theta = \frac{\pi}{9} \) to get:

\[ \frac{1}{2} = \cos\left(\frac{\pi}{3}\right) = 4\cos^3\left(\frac{\pi}{9}\right) - 3\cos\left(\frac{\pi}{9}\right). \]

\[ 4x^3 - 3x - \frac{1}{2} = 0, \text{ where } x = \cos\left(\frac{\pi}{9}\right). \]

Let \( y = x/2 \) to get:

\[ \frac{4y^3 - 3y - \frac{1}{2}}{8} = 0 \]

\[ y^3 - 3y - 1 = 0 \text{ where } y = \frac{\cos\left(\frac{\pi}{9}\right)}{2}. \]

Claim: \( y^3 - 3y - 1 = 0 \) has no solution in \( \mathbb{Q} \).

Proof: Suppose \( y = \frac{a}{b} \) is a solution with \( a, b \in \mathbb{Z} \), \( a, b \) coprime (no common factor).

Then \[ \frac{a^3}{b^3} - \frac{3a}{b} - 1 = 0 \]

\[ \frac{a^3 - 3ab^2 - b^3}{b^3} = 0. \]

\[ a(a^2 - 3b^2) = b^3. \]

\[ \Rightarrow a \mid b^3. \text{ But } a, b \text{ coprime.} \]

Hence \( a = \pm 1 \).
Similarly \( a^3 = b(3a + b^2) \)

\[\Rightarrow b \mid a^3 \Rightarrow b = \pm 1.\]

So the only possible \( \mathbb{R} \) roots are \( \pm \frac{1}{\pm 1} \).

But

\[
\begin{align*}
(+1)^3 - 3(+1) - 1 & \neq 0 \\
(-1)^3 - 3(-1) - 1 & \neq 0
\end{align*}
\]

Corollary: \( \cos \left( \frac{\pi}{3} \right) \notin \mathbb{Q} \)

General method.

**Rational Root Test:**

Given \( f(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 \in \mathbb{Z}[x] \),

If \( f\left( \frac{a}{b} \right) < 0 \) for \( \frac{a}{b} \in \mathbb{Q} \), then

\[ a \mid c_0 \text{ AND } b \mid c_n. \]

\( \Rightarrow \) Finitely many possibilities that can be checked.

*eg.* \( 3x^3 - 5x^2 + 5x - 2 = 0 \).

**Q - roots restricted to** \( \pm 1, \pm 2, \pm \frac{1}{3}, \pm \frac{2}{3} \)

\( \pm 1, \pm \frac{2}{3} \) plug them in.
Lemma: Consider \( \alpha, F, E \), \( F \subseteq F[\sqrt{E}] \) with conjugation at \( c : \delta \rightarrow a - b \sqrt{c} \).

If cubic \( f(x) \in F[x] \) has a root in \( F[\sqrt{c}] \), then it also has a root in \( F \).

Proof: Suppose \( \alpha \in F[\sqrt{c}] \) with \( f(\alpha) = 0 \).

If \( \alpha \in F \) done, otherwise note that

\[
f(\alpha) = 0
\]

\[
f(\overline{\alpha}) = 0
\]

So \( \overline{\alpha} \) is another root \((\alpha \neq \overline{\alpha} \text{ since } \alpha \notin F) \).

Factor to get

\[
f(x) = (x - \alpha)(x - \overline{\alpha})(x - \beta)
\]

for some \( \beta \).

\[
= (x^2 - (\alpha + \overline{\alpha})x + \alpha \overline{\alpha})(x - \beta).
\]

But note \( \alpha + \overline{\alpha} \in F \)

\( \alpha \overline{\alpha} \in F \).

Hence \( \beta = \frac{f(\alpha)}{x^2 - (\alpha + \overline{\alpha})x + \alpha \overline{\alpha}} \in F \), all in \( F[x] \).
Theorem: \( \cos \left( \frac{\pi}{9} \right) \) is NOT constructible.

Proof: Suppose (for contradiction) that it is constructible.

Then \( \mathbb{Q} \) chain of \( \mathbb{Q}(\sqrt{3}) \):

\[ \mathbb{Q} = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_k \subseteq \cdots \subseteq \mathbb{R} \]

such that \( \cos \left( \frac{\pi}{9} \right) \in F_k \).

But then \( x^3 - 3x - 1 \) has a root in \( F_k \).

Hence it has a root in \( F_{k-1} \) (by lemma)...

\[ \cdots \subseteq \cdots \subseteq \in F_{k-2} \]

Hence it has a root in \( F_0 = \mathbb{Q} \).

But \( x^3 - 3x - 1 \) has no \( \mathbb{Q} \)-root.

\[ \square \]

Trisecting an Angle is Impossible! (in general).