A.1. **Euclid’s Lemma.** Suppose that $a$ divides $bc$ for $a, b, c \in \mathbb{Z}$ with $a$ and $b$ coprime (i.e. they have no common factor except ±1). **Prove** that $a$ must divide $c$. (Hint: Since $a$ and $b$ are coprime, you may assume — without proof — that there exist $x, y \in \mathbb{Z}$ such that $ax + by = 1$.)

A.2. **Prove** that $\sqrt{2}$ is not rational.

A.3. Consider a quadratic field extension $F \subseteq F[\sqrt{c}] = \{a + b\sqrt{c} : a, b \in F\}$ and define the conjugation map $a + b\sqrt{c} \mapsto a - b\sqrt{c}$. **Prove** that for all $u, v \in F[\sqrt{c}]$ we have

- $\overline{u + v} = \overline{u} + \overline{v}$,
- $\overline{uv} = \overline{u} \overline{v}$.

A.4. Consider again the same field extension $F \subseteq F[\sqrt{c}]$ and let $p(x) \in F[x]$ be a polynomial with coefficients in $F$. **Prove** that for any $\alpha \in F[\sqrt{c}]$ we have

$$p(\alpha) = 0 \iff p(\overline{\alpha}) = 0.$$  

For the next two problems you may assume — without proof — that $2 \cos(2\pi/7)$ is a root of $x^3 + x^2 - 2x - 1 = 0$.

A.5. **Prove** that $x^3 + x^2 - 2x - 1 = 0$ has no rational root, and hence that $\cos(2\pi/7)$ is not rational.

A.6. **Prove** that $\cos(2\pi/7)$ is not constructible, and hence that the regular heptagon is not constructible with straightedge and compass.

Note: We have now proved that the following classical problems are impossible: “doubling the cube”, “trisecting an angle”, “constructing the regular heptagon”. The only problem left is “squaring the circle”, which is equivalent to constructing $\pi$. Lindemann (1882) proved that $\pi$ is not constructible, but I’m not clever enough to present the proof to you. (Wikipedia has it.)