Math 461 F | **Spring 2011**
Homework 4 | **Drew Armstrong**

Reading.

Section 2.4

Problems.

A.1. Euclid's Lemma. Suppose that a divides bc for $a, b, c \in \mathbb{Z}$ with a and b coprime (i.e. they have no common factor except ± 1). **Prove** that a must divide c. (Hint: Since a and b are coprime, you may assume — without proof — that there exist $x, y \in \mathbb{Z}$ such that $ax + by = 1$.)

A.2. Prove that $\sqrt[3]{2}$ is not rational.

A.3. Consider a quadratic field extension $F \subseteq F[\sqrt{c}] = \{a + b\sqrt{c} : a, b \in F\}$ and define the conjugation map $a + b\sqrt{c} \mapsto a - b\sqrt{c}$. **Prove** that for all $u, v \in F[\sqrt{c}]$ we have

- $\overline{u+v} = \overline{u} + \overline{v}$,
- $\overline{uv} = \overline{u}\,\overline{v}$.

A.4. Consider again the same field extension $F \subseteq F[\sqrt{c}]$ and let $p(x) \in F[x]$ be a polynomial with coefficients in F. Prove that for any $\alpha \in F[\sqrt{c}]$ we have

$$p(\alpha) = 0 \quad \Longleftrightarrow \quad p(\overline{\alpha}) = 0.$$

For the next two problems you may assume — without proof — that $2\cos(2\pi/7)$ is a root of $x^3 + x^2 - 2x - 1 = 0$.

A.5. Prove that $x^3 + x^2 - 2x - 1 = 0$ has no rational root, and hence that $\cos(2\pi/7)$ is not rational.

A.6. Prove that $\cos(2\pi/7)$ is not constructible, and hence that the regular heptagon is not constructible with straightedge and compass.

Note: We have now proved that the following classical problems are impossible: "doubling the cube", "trisecting an angle", "constructing the regular heptagon". The only problem left is "squaring the circle", which is equivalent to constructing π. Lindemann (1882) proved that π is not constructible, but I'm not clever enough to present the proof to you. (Wikipedia has it.)