Math 461 F — **Spring 2011**

Homework 3 — **Drew Armstrong**

Reading.

Section 2.4

Problems.

A.1. How many different (complex) numbers does the expression $\sqrt{1+\sqrt{3}}$ represent? Find a polynomial over $\mathbb{Z}$ which has these numbers as its roots.

A.2. Use the trigonometric identity $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$ together with Cardano's formula to find an expression for $\cos(\pi/9)$. (Note: This expression **must** involve complex numbers because $\cos(\pi/9)$ is not constructible.)

A.3. Suppose that $p = 2^a + 1$ is a prime number. Show that a must be a power of 2. (Hint: If a has an **odd** factor b, show that the polynomial $x^b + 1$ factors nicely.)

A.4. Prove that

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}.$$

(You can assume that the expression on the right converges.) We can describe this process **recursively** by setting $s_0 = 1$ and $s_n = 1 + 1/(1 + s_{n-1})$ for $n \geq 1$. What is s_4? How close is this to $\sqrt{2}$?

Let D be the set of numbers that can be formed from $1, +, -, \times, \div, \sqrt{\ }$ in a finite number of steps. Now suppose that (x, y) is an intersection point for two circles

$$(x-a)^2 + (y-b)^2 = R^2, \tag{1}$$

$$(x-c)^2 + (y-d)^2 = r^2, \tag{2}$$

where $a, b, c, d, r, R \in D$. We want to show that $x, y \in D$.

A.5. What change of variables $(x, y) \to (x', y')$ **translates** the plane by (a, b) and **then rotates** the plane by angle $\tan^{-1}((b+d)/(a+c))$? Observe that $x', y' \in D \Leftrightarrow x, y \in D$.

A.6. Applying this transformation sends the center of circle (1) to the origin and then rotates the center of circle (2) to the x-axis, yielding a new system

$$x'^2 + y'^2 = R^2, \tag{3}$$

$$(x' - \alpha)^2 + y'^2 = r^2, \tag{4}$$

where $\alpha \in D$. Show that $x', y' \in D$, and hence $x, y \in D$.