Math 461 F — Spring 2011
Exam 2 — Fri Mar 25 — Drew Armstrong

Problem 1. [6 points]
(a) **Accurately state** Gauss and Wantzel's theorem on the constructibility of regular polygons with straightedge-and-compass.

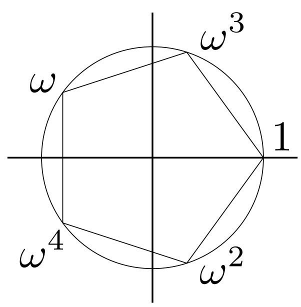
A regular n-gon is constructible **if and only if** n is equal to a power of 2 multiplied by **distinct** Fermat primes.

(b) **Yes or no.** For the following values of n, **state** whether the regular n-gon is constructible.

n	regular n-gon constructible?
5	Yes
7	No
15	Yes
17	Yes

Problem 2. [6 points] In this problem we want to compute $\cos\left(\frac{4\pi}{5}\right)$.

(a) Let $\omega = \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right)$. **Label the vertices** of the given regular pentagon (in the complex plane) by powers of ω.



(b) **Find a formula** for $u = \omega + \omega^{-1}$ and **solve it** to find $\cos\left(\frac{4\pi}{5}\right)$. (Hint: The sum of the five vertices is zero.)

Since $u = \omega + \omega^{-1} = \omega + \overline{\omega} = 2\cos\left(\frac{4\pi}{5}\right)$, we wish to solve for u. We know that the sum of all of the fifth roots of unity is zero. That is,

$$\omega^2 + \omega + 1 + \omega^{-1} + \omega^{-2} = 0.$$

So we wish to express the sum of these roots in terms of u. First note that $u^2 = (\omega + \omega^{-1})^2 = \omega^2 + 2\omega\omega^{-1} + \omega^{-2} = \omega^2 + 2 + \omega^{-2}$. Hence

$$u^2 + u - 1 = \omega^2 + \omega + 1 + \omega^{-1} + \omega^{-2} = 0.$$

By the quadratic formula we have $u = \frac{-1\pm\sqrt{5}}{2}$. Since $2\cos(\frac{4\pi}{5})$ is **negative**, we choose the negative root to get $2\cos(4\pi/5) = (-1-\sqrt{5})/2$, or

$$\cos(4\pi/5) = \frac{-1-\sqrt{5}}{4}.$$

Problem 3. [6 points]

(a) Let a, b be positive integers. **Factor** $2^{ab} - 1$ as a product of two integers.

First recall the general formula for a difference of like powers:

$$x^b - 1 = (x-1)(1 + x + x^2 + \cdots + x^{b-1}).$$

Now substitute 2^a into this expression to observe that the integer $2^{ab} - 1 = (2^a)^b - 1$ factors as the product of two integers

$$(2^a - 1)(1 + 2^a + 2^{2a} + \cdots + 2^{(b-1)a}).$$

(b) Let $n > 1$ be a positive integer and **prove** the following:

If $2^n - 1$ is prime then n is prime.

Proof. Let $2^n - 1$ be a prime integer and suppose (for contradiction) that n is **not** prime. Thus we can write $n = ab$ where a, b are integers both greater than 1. But then part (a) implies that

$$2^n - 1 = (2^a - 1)(1 + 2^a + 2^{2a} + \cdots + 2^{(b-1)a}).$$

Since a, b are both greater than 1 we see that the two factors of $2^n - 1$ are both greater than 1. This **contradicts** the fact that $2^n - 1$ is prime. Hence n must be prime. □

Problem 4. [6 points] Let $f(x) \in \mathbb{Q}[x]$ be a **cubic (i.e. degree** 3**)** polynomial with rational coefficients, such that $f(1 + \sqrt{2}) = 0$.

(a) Explain why $f(x) = (x^2 - 2x - 1)g(x)$ for some $g(x) \in \mathbb{Q}[x]$ of degree 1. (You may use any result from class without proof.)

Since $1 + \sqrt{2}$ is a root, we know that its conjugate (in $\mathbb{Q}[\sqrt{2}]$) is also a root. Hence by the Factor Theorem we can write

$$\begin{aligned} f(x) &= \left(x - \left(1 + \sqrt{2}\right)\right)\left(x - \left(1 - \sqrt{2}\right)\right)g(x), \\ &= (x^2 - 2x - 1)g(x), \end{aligned}$$

where $g(x)$ has degree 1. The coefficients of $g(x)$ are rational since otherwise expanding the right hand side would show that $f(x)$ has non-rational coefficients, a contradiction.

(b) **Prove** that $f(x)$ has a rational root. (Hint: Prove that $g(x)$ has a rational root.)

Since $g(x)$ has rational coefficients and degree 1, we can write $g(x) = ax + b$ with $a, b \in \mathbb{Q}$ and $a \neq 0$. This implies that $g(-b/a) = 0$, and hence

$$f(-b/a) = ((-b/a)^2 - 2(-b/a) - 1)g(-b/a) = 0.$$

We conclude that $f(x)$ has a rational root; namely, $-b/a$.

Statistics: 41 exams were submitted. The Average/Median/Standard Deviation were 17.49, 18, and 4.57, respectively. Three students received 24/25 and one student received 25/25.