Math 461 F — Spring 2011
Exam 1 — Fri Feb 18 — Drew Armstrong

The exam is worth 25 points. There are 4 problems worth 6 points each, and you get 1 point for writing your name.

Problem 1. [6 points] Consider the cubic equation $ax^3 + bx^2 + cx + d = 0$.

(a) **What change of variable** do you make to get the "depressed" cubic equation?

You set $x = y - b/3a$ to get an equation in y with no y^2 term. (This is called the "depressed" cubic.) You can divide the whole equation by a if you want, but this is not so important (I gave full points either way.)

(b) Now **suppose** that the depressed equation has roots $-1, 0, 1$. In this case, **what are the roots of the original?**

If $y = -1, 0, 1$ are the roots of the depressed cubic, then the original has roots

$$x = -1 - \frac{b}{3a}, \qquad x = 0 - \frac{b}{3a}, \qquad x = 1 - \frac{b}{3a}.$$

(c) Use the information from (b) to **completely factor** $ax^3 + bx^2 + cx + d$.

Since we know the roots, we can factor the polynomial $ax^3 + bx^2 + cx + d$ to get

$$a\left(x + 1 + \frac{b}{3a}\right)\left(x + \frac{b}{3a}\right)\left(x - 1 + \frac{b}{3a}\right).$$

Problem 2. [6 points]

(a) **Use de Moivre's formula to prove** that $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$.

First we expand de Moivre's formula to get

$$\begin{aligned}
\cos(3\theta) + i\sin(3\theta) &= (\cos\theta + i\sin\theta)^3 \\
&= \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta \\
&= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\sin\theta\cos^2\theta - \sin^3\theta).
\end{aligned}$$

Now we equate the *real* parts and use $\cos^2\theta + \sin^2\theta = 1$ to get

$$\begin{aligned}
\cos(3\theta) &= \cos^3\theta - 3\cos\theta\sin^2\theta \\
&= \cos^3\theta - 3\cos\theta(1 - \cos^2\theta) \\
&= \cos^3\theta - 3\cos\theta + 3\cos^3\theta \\
&= 4\cos^3\theta - 3\cos\theta,
\end{aligned}$$

as desired.

(b) **Find a similar formula** for $\sin(3\theta)$ in terms of $\sin\theta$.

If we equate the *imaginary* parts of the formula, we get

$$\begin{aligned}\sin(3\theta) &= 3\sin\theta\cos^2\theta - \sin^3\theta \\ &= 3\sin\theta(1-\sin^2\theta) - \sin^3\theta \\ &= 3\sin\theta - 3\sin^3\theta - \sin^3\theta \\ &= 3\sin\theta - 4\sin^3\theta.\end{aligned}$$

Problem 3. [6 points]

(a) **Find the complete solution** to $x^4 + 4 = 0$, or $x^4 = -4$.

(There are several ways to do this problem; your solution may look different.) Note that -4 has modulus 4 and angle π. Thus, if $x^4 = -4$, then x must have modulus $\sqrt[4]{4} = \sqrt{2}$ (here we take the **real, positive** square root) and angle θ where $4\theta = \pi + 2\pi k$ for any $k \in \mathbb{Z}$. There are four such angles:

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.$$

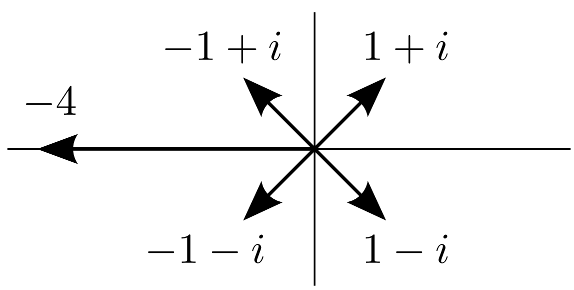
We conclude that the solutions are

$$\sqrt[4]{-4} = \left\{\sqrt{2}\,e^{\pi i/4}, \sqrt{2}\,e^{3\pi i/4}, \sqrt{2}\,e^{5\pi i/4}, \sqrt{2}\,e^{7\pi i/4}\right\}.$$

Since we know $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$, these roots can be simplified to get

$$\sqrt[4]{-4} = \{1+i, -1+i, -1-i, 1-i\}.$$

Here's a picture.



(b) **Factor** $x^4 + 4$ into two quadratics with **real coefficients**.

Now recall that for $z = a + ib$ we have $(x-z)(x-\bar{z}) = x^2 - 2ax + |z|^2$, which is real. Using this idea we group the roots into conjugate pairs to get

$$\begin{aligned}x^4 + 4 &= [(x-(1+i))(x-(1-i))]\,[(x-(-1+i))(x-(-1-i))] \\ &= (x^2 - 2x + (1^2+1^2))(x^2 - (-2)x + (1^2+1^2)) \\ &= (x^2 - 2x + 2)(x^2 + 2x + 2).\end{aligned}$$

Problem 4. [6 points] Consider the cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$.

(a) **Compute a formula** for $\dfrac{f(x) - f(y)}{x - y}$.

We have

$$\begin{aligned} f(x) - f(y) &= (ax^3 + bx^2 + cx + d) - (ay^3 + by^2 + cy + d) \\ &= a(x^3 - y^3) + b(x^2 - y^2) + c(x - y) \\ &= a(x - y)(x^2 + xy + y^2) + b(x - y)(x + y) + c(x - y) \\ &= (x - y)\left[a(x^2 + xy + y^2) + b(x + y) + c\right], \end{aligned}$$

or

$$\frac{f(x) - f(y)}{x - y} = a(x^2 + xy + y^2) + b(x + y) + c.$$

(b) Now **suppose** that $f(1) = 0$. In this case, **use your formula** to factor $f(x)$.

Putting $y = 1$ into the formula yields

$$f(x) = f(x) - f(1) = (x - 1)\left[a(x^2 + x + 1) + b(x + 1) + c\right].$$

Note: I did not ask you to prove Descartes' Factor Theorem, but Problem 4 tested your understanding of the proof. Problem 1 tested your understanding of why we care about the depressed cubic. Problem 2 tested your understanding of de Moivre's formula. Problem 3 tested your understanding of roots of unity and conjugates. (And once upon a time I assigned Problem 3(a) as an exercise in class.)

The **average** for this exam was 17.79/25 and the **median** was 19/25. Out of 43 students, 10 students received a score of 24 or 25 out of 25. I do **not** assign letter grades for exams, but I estimate the following **approximate** grade ranges:

$$\begin{aligned} 20 - 25 &\approx A, \\ 15 - 19 &\approx B, \\ 9 - 14 &\approx C. \end{aligned}$$