This Week: Chapter 5

The mathematical core of Calculus is differentiation and integration of functions. Here is a summary:

Kind of function	diff	int
$\mathbb{R} \to \mathbb{R}$	Calc I	Calc I & II
$\mathbb{R} \to \mathbb{R}^n$	Ch 1 & 3	Ch 3 & 6
$\mathbb{R}^n \to \mathbb{R}$	Ch 4	Ch 5
$\mathbb{R}^m \to \mathbb{R}^n$	Ch 6	Ch 6

DONE.

NOW.

We can't cover this completely.

Chapter 5: Integration of scalar fields in $\mathbb{R}^2$ & $\mathbb{R}^3$.

Example: Consider $f(x,y) = xy^2$.

Think of this as the height of a surface above x,y-plane:

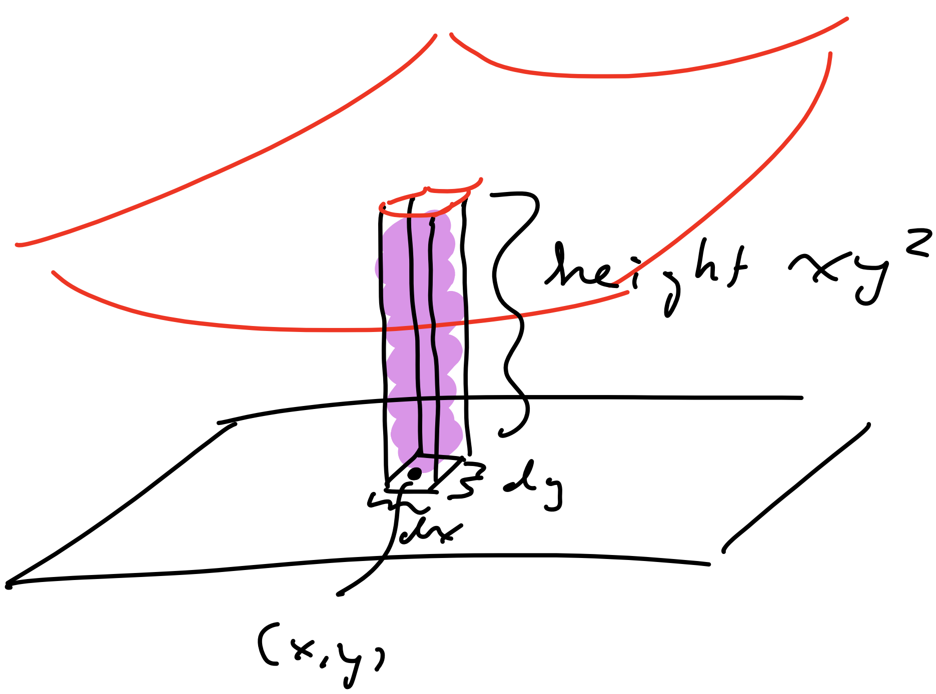
[see Geogebra].

Compute the volume of solid region above the square

$$0 \leq x \leq 1$$
$$0 \leq y \leq 1$$

and below the surface.

Idea: Consider a skinny vertical column above the point (x,y):



volume of skinny column

= height × area of base

= $xy^2\,dx\,dy$.

To obtain volume of the region, "add up all the skinny columns":

$$\text{volume} = \iint xy^2\,dx\,dy$$

how to compute?

It's just two integrals, one with respect to x & one with respect to y, and the <u>order doesn't matter</u>. Let's integrate x first.

$$\text{vol} = \int_{y=0}^{y=1} \left(\int_{x=0}^{x=1} xy^2\,dx \right) dy$$

$$= \int_{y=0}^{y=1} \left[\frac{1}{2} x^2 y^2 \right]_0^1 dx$$

$$= \int_{y=0}^{y=1} \left(\frac{1}{2} y^2 - 0 \right) dy$$

$$= \int_0^1 \frac{1}{2} y^2 \, dy$$

$$= \left[\frac{1}{2} \cdot \frac{1}{3} y^3 \right]_0^1$$

$$= \frac{1}{6} - 0 = 1/6$$

The exact volume of the 3D region is $1/6$.

Check that order doesn't matter:

Now do y first:

$$\text{vol} = \int_{x=0}^{x=1} \left(\int_{y=0}^{y=1} x y^2 \, dy \right) dx$$

$$= \int_{x=0}^{x=1} \left[\frac{1}{3} x y^3 \right]_{y=0}^{y=1} dx$$

$$= \int_{x=0}^{x=1} \left(\frac{1}{3} x - 0 \right) dx$$

$$= \int_0^1 \frac{1}{3} x \quad dx$$

$$= \left[\frac{1}{3} \cdot \frac{1}{2} x^2 \right]_{x=0}^{x=1}$$

$$= \frac{1}{6} - 0 = 1/6 \quad \checkmark$$

Integrate over a different region:

$$\int_{y=0}^{y=1} \left(\int_{x=-1}^{x=1} x y^2 \, dx \right) dy$$

$$= \int_{y=0}^{y=1} \left(\frac{1}{2} x^2 y^2 \right)_{x=-1}^{x=1} dy$$

$$= \int_0^1 \left(\frac{1}{2}y^2 - \frac{1}{2}y^2\right) dy$$

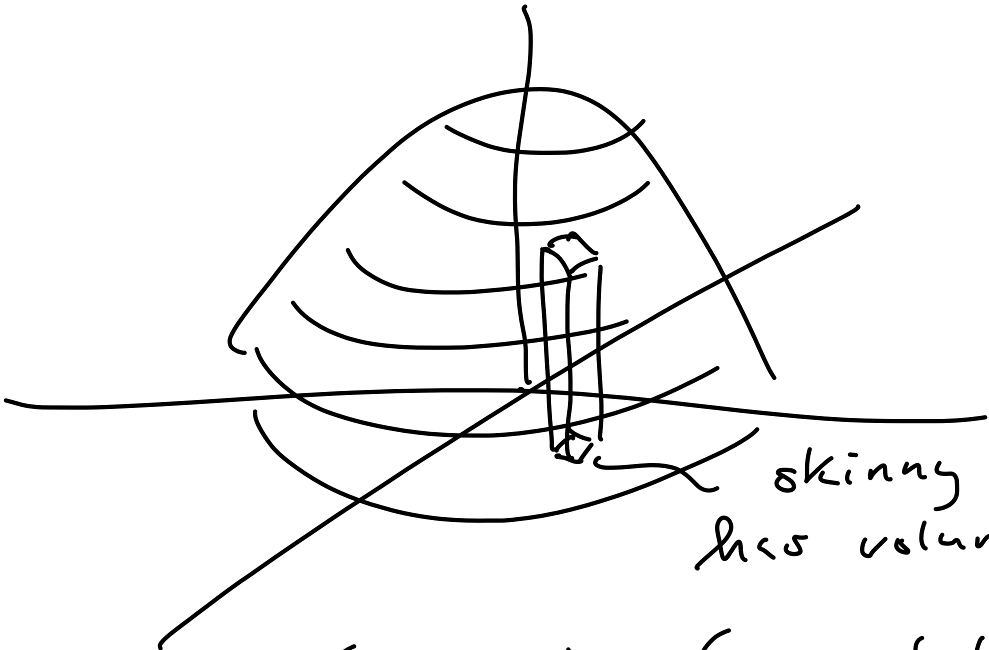
$$= \int_0^1 0 \, dy = 0.$$

So the "volume" of 3D region between rectangle

$$-1 \leq x \leq +1$$
$$0 \leq y \leq 1$$

and the surface $z = xy^2$ is zero. How can a volume be zero? Volume BELOW the x,y plane counts as negative.

Harder Example: Compute volume between x,y plane & parabolic dome $z = 1 - x^2 - y^2$:



(height) $\times$ (area of base)

$(1-x^2-y^2)\,dx\,dy$.

$$\text{Volume} = \iint (1-x^2-y^2)\,dx\,dy.$$

limits of integration?

Need to sum over all points (x,y) in the unit disk.

$$? \leq x \leq ?$$

$$? \leq y \leq ?$$

There are 2 ways to do it:

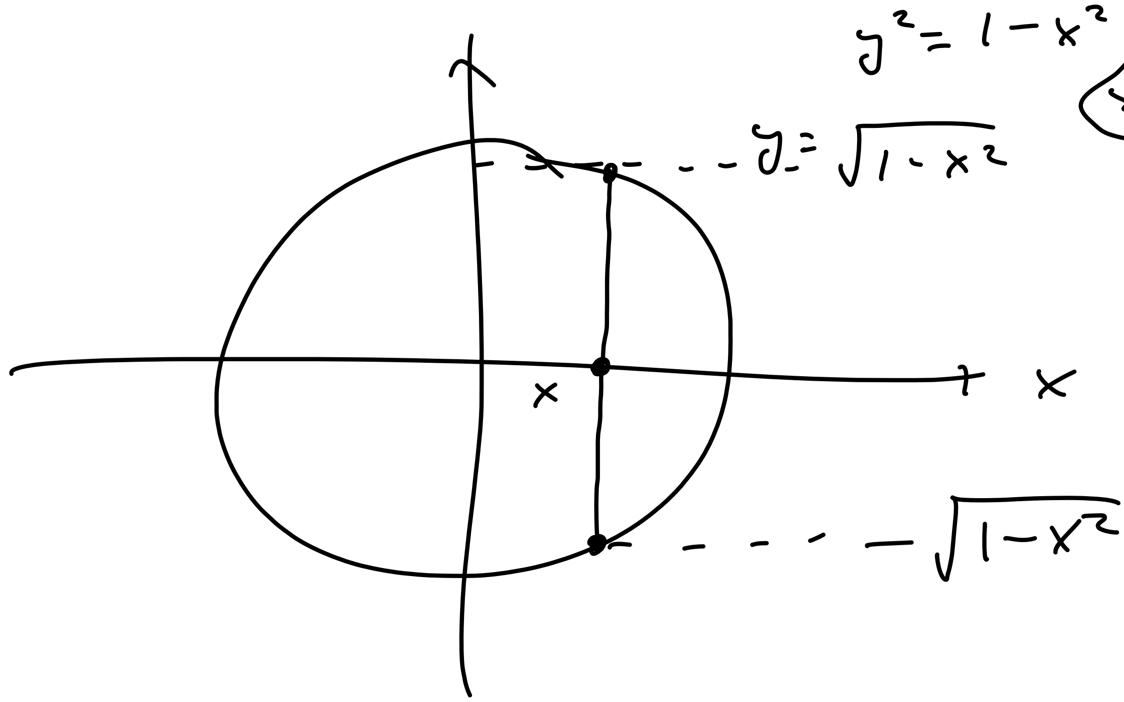
First let $-1 \leq x \leq +1$.

Then for each value of x, let

$$-\sqrt{1-x^2} \leq y \leq +\sqrt{1-x^2}$$

$y^2 = 1-x^2$

$x^2+y^2=1$



This choice of parametrization forces the order of integration:

$$\text{vol} = \int_{x=-1}^{1} \left(\int_{y=-\sqrt{1-x^2}}^{y=+\sqrt{1-x^2}} (1-x^2-y^2)\, dy \right) dx$$

$$= \int_{x=-1}^{1} \left[y - x^2 y - \frac{1}{3} y^3 \right]_{y=-\sqrt{1-x^2}}^{y=+\sqrt{1-x^2}} dx$$

$$= 2 \int (1-x^2)\sqrt{1-x^2} - \frac{1}{3}\left(\sqrt{1-x^2}\right)^3 \, dx$$

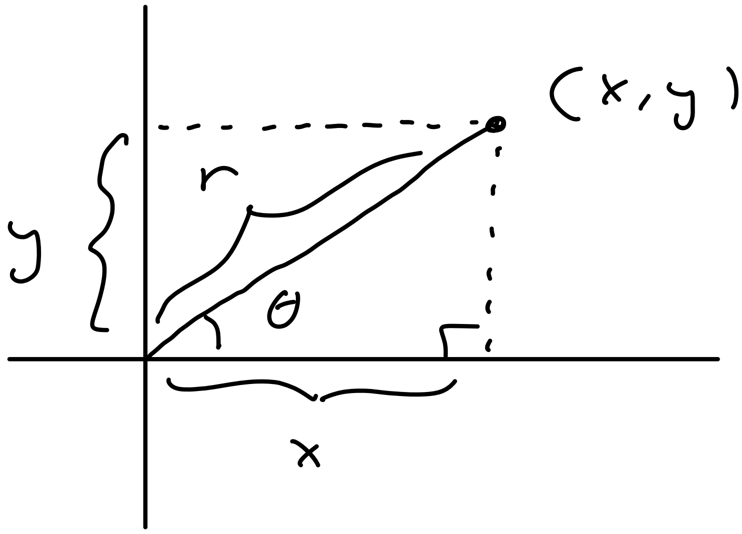
$$= 2 \int_{x=-1}^{+1} \frac{2}{3}(1-x^2)^{3/2} \, dx$$

This is bad & my compter says answer is $\pi/2$.

Since the answer is nice, there must be an easier way to do this.

Polar / Cylindrical Coordinates.

When we integrate over a region of x,y plane with rotational symmetry it's better to use polar coordinates:

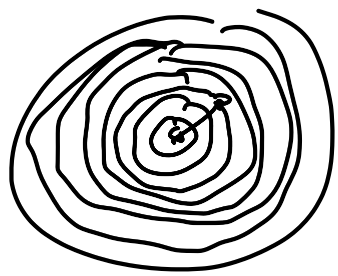


$$x = r\cos\theta \qquad x^2+y^2 = r^2$$
$$y = r\sin\theta$$

just constants

Parametrize the unit disk:

$$0 \le r \le 1$$
$$0 \le \theta \le 2\pi.$$



Nice property of $z = (1-x^2-y^2)$.

$$\begin{aligned} x^2+y^2 &= r^2\cos^2\theta + r^2\sin^2\theta \\ &= r^2\left(\cancel{\cos^2\theta + \sin^2\theta}\right)\ 1 \\ &= r^2 \end{aligned}$$

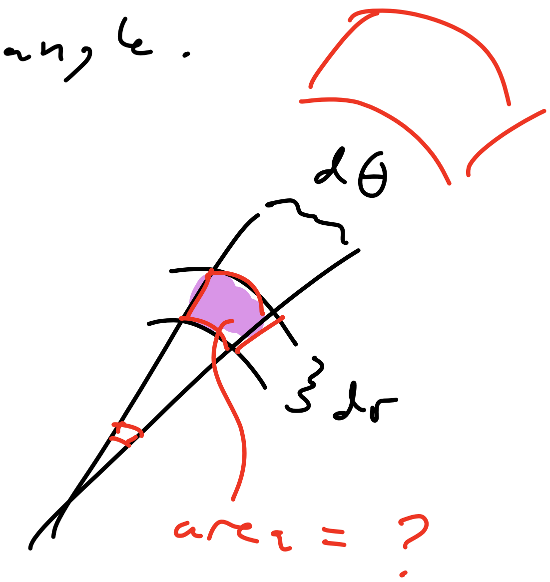
So $\quad z = (1-x^2-y^2) = 1-r^2.$

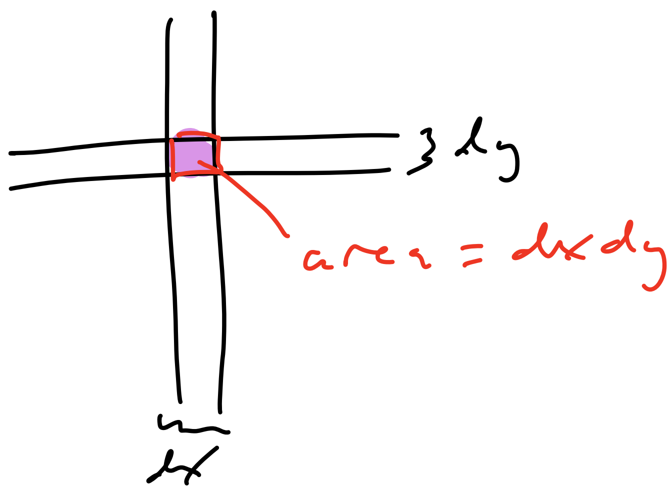
Idea:

$$\text{vol} = \int_{\theta=0}^{2\pi}\int_{r=0}^{1}(1-r^2)\,dr\,d\theta\,.$$

But this is not quite correct!

Reason: For tiny changes in r & θ we don't get a rectangle.

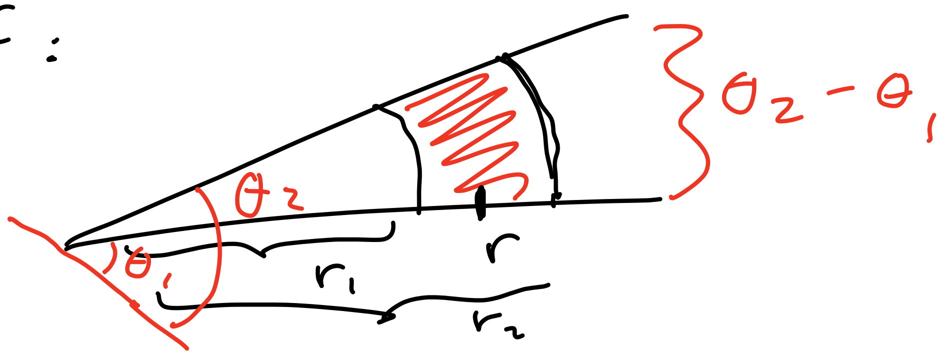




Theorem: Tiny region caused by tiny changes in r & θ has area

$$r\,dr\,d\theta\,.$$

Fake Proof:



Area:

$$\underbrace{(\pi r_2^2 - \pi r_1^2)}_{\text{area between circles}} \underbrace{(\theta_2 - \theta_1)/2\pi}_{\text{how much of the circle do you want.}}$$

Let $r_2 \to r_1$ & $\theta_2 \to \theta_1$

$r_2 - r_1 \to dr$ $\qquad \theta_2 - \theta_1 = d\theta$

$r_2, r_1 \longrightarrow r.$

Area:

$$\frac{1}{2}(r_2^2 - r_1^2)(\theta_2 - \theta_1)$$

$$= \frac{1}{2}\underbrace{(r_1 + r_2)}_{r+r}\underbrace{\cancel{(r_2 - r_1)}}_{dr}\underbrace{\cancel{(\theta_2 - \theta_1)}}_{d\theta}$$

$$= r\,dr\,d\theta.$$

As I said, it's a fake proof.

Just a heuristic.

Correct Computation:

vol of parabolic dome

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \underbrace{(1-r^2)}_{\text{height of skinny column}} \underbrace{r\,dr\,d\theta}_{\text{area of base of skinny column.}}$$

$$= \int_{\theta=0}^{2\pi} \left(\int_{r=0}^{1} (r-r^3)\,dr \right) d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(\frac{1}{2}r^2 - \frac{1}{4}r^4 \right)\Bigg|_{r=0}^{r=1} d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(\frac{1}{2} - \frac{1}{4} \right) d\theta$$

$$= \int_0^{2\pi} \frac{1}{4}\,d\theta = \frac{1}{4}\cdot 2\pi = \frac{\pi}{2} \checkmark$$

Chapter 5 : Integration in 2D & 3D.

Recall: Given a scalar field $f(x,y)$ in $\mathbb{R}^2$ and a 2D region $D \subseteq \mathbb{R}^2$ (e.g. rectangle, circle, ..)

"is a subset of"

Then we can integrate $f(x,y)$ over D:

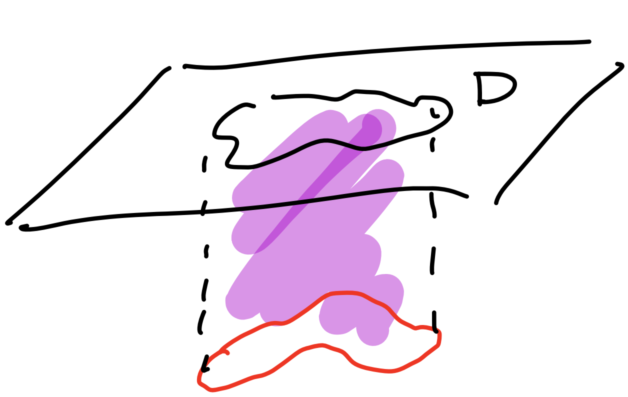
$$I = \iint_D f(x,y)\,dx\,dy = \text{a scalar}$$

Two possible interpretations:

- $f(x,y)$ = height of a surface "above" the x,y plane. Then

  I = "signed volume" of 3D region "above" D and "below" the surface.

I = volume.



$I = -$(volume)

- $f(x,y)$ = mass density
  = mass / unit area.

Then

$$I = \iint_D \underbrace{\overbrace{f(x,y)}^{\text{density}}\,\overbrace{dx\,dy}^{\text{tiny area}}}_{\text{mass of tiny piece}} = \text{total mass of 2D region } D.$$

[Total Mass $= \sum$ point masses

$= \int$ continuous density.]

Can also use this interpretation to compute area. If

density $= 1$ unit / unit area.

Then

$$\text{area}(D) = \text{total mass}$$

$$= \iint_D 1 \, dx\,dy$$

Integration over rectangles is "easy".

Consider rectangle

$$R = [a_1, a_2] \times [b_1, b_2]$$

= the set of points $(x,y) \in \mathbb{R}^2$

where $a_1 \leq x \leq a_2$ & $b_1 \leq y \leq b_2$.

$$\iint_R f(x,y)\,dxy$$

$$= \int_{y=b_1}^{b_2} \left(\int_{x=a_1}^{a_2} f(x,y)\,dx \right) dy$$

SAME (Fubini's Theorem)

$$= \int_{x=a_1}^{a_2} \left(\int_{y=b_1}^{b_2} f(x,y)\,dy \right) dx$$

[ASIDE: Surface area.

Parametrized surface in 3D.

$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$.

Let D be a 2D region in the curved surface. Then

$$\text{area}(D) = \iint_D \underbrace{\|\vec{r}_u \times \vec{r}_v\|\,du\,dv}_{\text{area of a tiny piece of surface}} .$$

]

Parametrizing a rectangle is easy, but the resulting integral might still be hard.

TRICK: "u-substitution in 2D"

First Example: Polar Coordinates.

$$x = r\cos\theta \qquad r = \sqrt{x^2+y^2}$$

$$y = r\sin\theta \qquad \theta = \arctan(y/x)$$

(Going back is ugly)

Replace:

$$\underbrace{dx\,dy}_{\text{tiny piece of area}} = r\,\underbrace{dr\,d\theta}_{\text{tiny piece of area}}$$

Sometimes we just write dA for a tiny piece of area. Then we don't have to say what the coordinates are:

$$\iint_{D} F \, dA$$

tiny piece of area.

2D region

scalar field in 2D

Polar Coords work best when region D is a circle, or annulus, of sector of a circle, ...

We used it last time to solve

$$\iint_{x^2+y^2 \leq 1} (1-x^2-y^2)\, dx\, dy$$

$$= \iint_{\substack{0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi}} (1-r^2)\, r\, dr\, d\theta = \frac{\pi}{2}$$

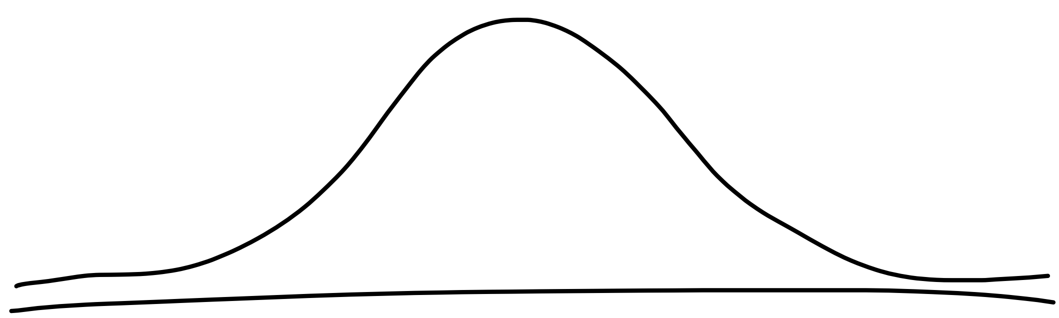
Another Example (Famous Trick):

The indefinite integral $\int e^{-x^2}\,dx$

does not have an elementary formula (i.e. cannot be expressed in terms of polynomials, roots, trig, log, exp). Nevertheless, the definite integral from $-\infty$ to ∞ has a (surprisingly) nice formula:

$$I = \int_{x=-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

Very important in statistics:



Normal ("Gaussian") distribution

Function $\frac{1}{\sqrt{\pi}} e^{-x^2}$ has total area 1, so it defines a "random variable".

Integral I is computed with a very clever trick:

$$I^2 = I \cdot I$$

$$= \int_{x=-\infty}^{\infty} e^{-x^2}\,dx \cdot \int_{y=-\infty}^{\infty} e^{-y^2}\,dy$$

$$= \iint_{\substack{\text{whole} \\ x,y \text{ plane}}} e^{-x^2} \cdot e^{-y^2}\,dx\,dy$$

$-r^2$ $\qquad$ $r^2 = x^2 + y^2$

$$= \iint e^{-x^2-y^2}\,dx\,dy$$

$r\,dr\,d\theta$

So far this looks silly. But now we change to polar coordinates.

$$= \iint_{\substack{\text{whole plane} \\ 0 \le r \le \infty \\ 0 \le \theta \le 2\pi}} e^{-r^2}\, r\, dr\, d\theta$$

Miracle! This can be integrated using "u-sub"

$$= \int_{\theta=0}^{2\pi} d\theta \cdot \int_{r=0}^{\infty} r e^{-r^2}\, dr$$

[Fact: $\iint f(r)\, g(\theta)\, dr\, d\theta$

$= \int g(\theta)\, d\theta \cdot \int f(r)\, dr.$]

$$= 2\pi \cdot \int_{r=0}^{\infty} r e^{-r^2}\, dr$$

$u = r^2$

$du = 2r\, dr \qquad dr = \frac{1}{2} r\, dr.$

$$= 2\pi \int_{u=0}^{\infty} \frac{1}{2} e^{-u} \, du$$

$$= 2\pi \left[-\frac{1}{2} e^{-u} \right]_{u=0}^{u=\infty}$$

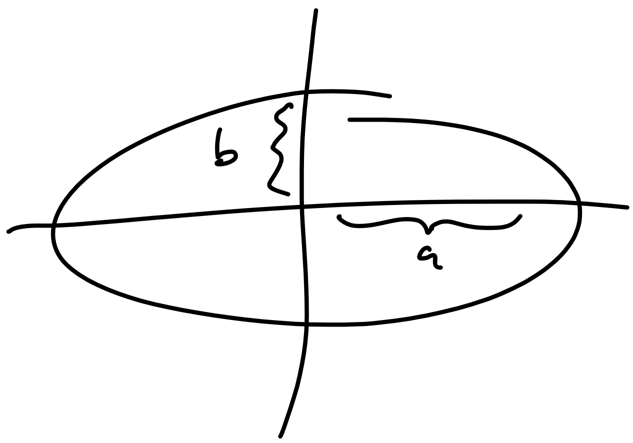
$$= 2\pi \left[-\frac{1}{2} \cancelto{0}{e^{-\infty}} + \frac{1}{2} \cancelto{1}{e^{0}} \right]$$

$$= 2\pi \left(\frac{1}{2} \right) = \pi \quad \checkmark$$

NICE!

Try to compute the area of an ellipse.

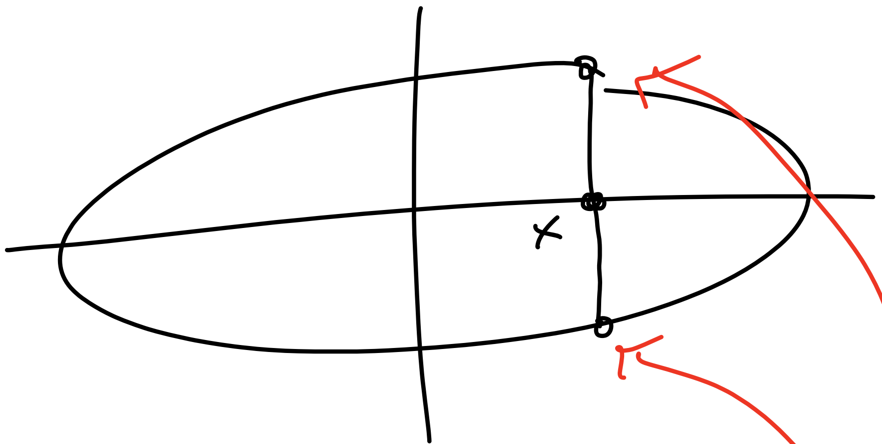
$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$



Let D be interior of ellipse.

$$\text{area}(D) = \iint_D dx\,dy.$$

How hard could it be?



$$-a \leq x \leq a$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$y = \pm b\sqrt{1 - x^2/a^2}$$

So

$$\text{area}(D) = \int_{x=-a}^{a} \left(\int_{y=-b\sqrt{1-\frac{x^2}{a^2}}}^{+b\sqrt{1-\frac{x^2}{a^2}}} 1\,dy \right) dx$$

LOOKS BAD!

TRY POLAR COORDS:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{r^2\cos^2\theta}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 1$$

$$r^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right) = 1.$$

Parametrize: $0 \leq \theta \leq 2\pi$

some bad function of θ $\leq r \leq$ some bad function of θ.

Problem: We really want to use

$$\cos^2\theta + \sin^2\theta = 1.$$

Seems to be a really easy idea.

Let $u = x/a$ & $v = y/b$.

Then

$$\text{area} = \iint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1} 1 \, dxy$$

$$= \iint_{u^2+v^2=1} 1 \, (du\,dv)^{?}$$

Question:

$$dx\,dy = \; ? \; du\,dv \, .$$

General Change of Coords in 2D
("u, v substitution")

Let $u(x,y)$ & $x(u,v)$
$v(x,y)$ $\quad$ $y(u,v)$

Chain Rule says

$$dx = \frac{dx}{du} \cdot du + \frac{dx}{dv} \cdot dv$$

$$dx = x_u \cdot du + x_v \cdot dv$$

$$dy = y_u \cdot du + y_v \cdot dv$$

Jacobian Matrix

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}$$

matrix multiplication

ROUGHLY

$$\text{"}dx\,dy\text{"} = \Big| \underbrace{\det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}}_{\text{area stretch factor.}} \Big| \text{"}du\,dv\text{"}$$

Example: Polar Coords

$x = r\cos\theta$ $\qquad$ $x_r = \cos\theta$

$y = r\sin\theta$ $\qquad$ $y_r = \sin\theta$

$x_\theta = -r\sin\theta$

$y_\theta = r\cos\theta$

$$dx\,dy = \left| \det \begin{pmatrix} x_r & x_\theta \\ y_r & y_\theta \end{pmatrix} \right| dr\,d\theta$$

$$= \left| \det \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \right| dr\,d\theta$$

$$= \left(r\cos^2\theta + r\sin^2\theta \right) dr\,d\theta$$

$$= r\,(\underbrace{\cos^2\theta + \sin^2\theta}_{1})\,dr\,d\theta$$

$$= r\,dr\,d\theta \quad \checkmark$$

This is the "real" way to do it.

Try to go backwards:

$$r = \sqrt{x^2+y^2}$$

$$\theta = \arctan(y/x)$$

$$dr\,d\theta = \left| \det \begin{pmatrix} r_x & r_y \\ \theta_x & \theta_y \end{pmatrix} \right| dx\,dy.$$

It should be $\frac{1}{r}$. You will check on HW 4 that it is.

[In general: Matrices

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \quad \& \quad \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$ are inverses.]

Back to the area of the ellipse

$$D: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Change coordinates

$u = x/a$ $\qquad$ $x = au$ $\qquad$ $x_u = a$, $x_v = 0$

$v = y/b$ $\qquad$ $y = bv$ $\qquad$ $y_u = 0$, $y_v = b$

$$dxdy = \left| \det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \right| dudv$$

[Mnemonic $dx = x_u\, du + x_v\, dv$]

$$\left[\ \left|\det\begin{pmatrix} x_u & y_u \\ x_v & y_v \end{pmatrix}\right| = \left|\det\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}\right|\ \right]$$

don't worry about columns & rows.

$$= \left|\det\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right| du\,dv$$

$$= ab\,du\,dv \,.$$

$$dx\,dy = ab\,du\,dv$$

HOW NICE!

Finally: Area of Ellipse:

$$\iint_{\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1} 1\,dx\,dy$$

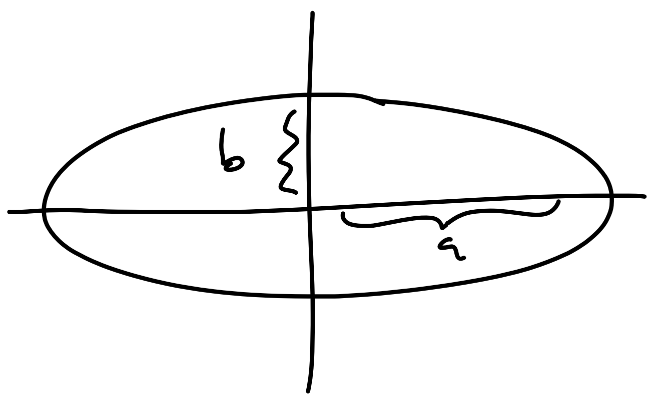
$$= \iint_{u^2+v^2=1} ab\,du\,dv\,.$$

$$= ab \iint_{u^2+v^2=1} 1 \, du \, dv$$

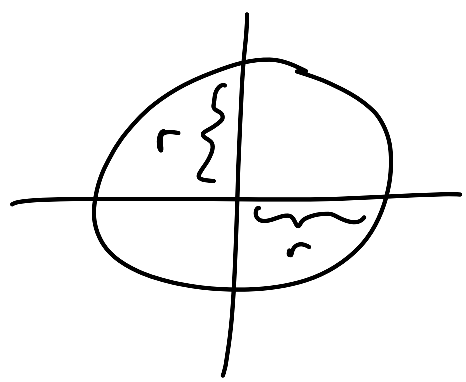
area of unit circle, $= \pi$

$$= \pi ab.$$

Compare to area of circle:



area $= \pi ab$



area $= \pi r^2$.

[Remark: Perimeter is much harder. Perimeter of an ellipse is a totally new kind of function.]

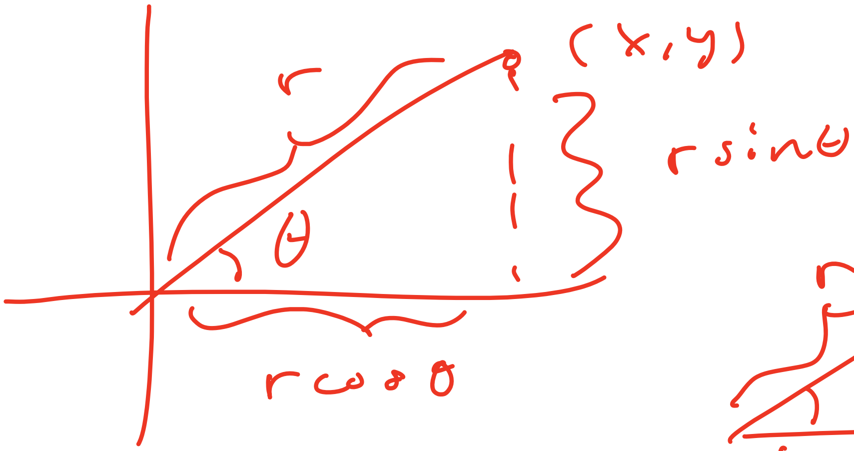
Same ideas work in 3D.

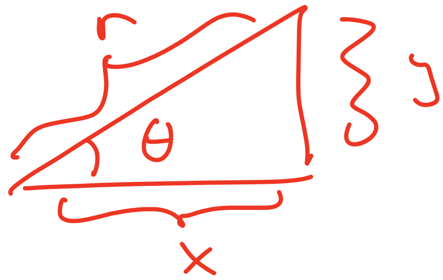
$u(x,y,z)$ $\qquad$ $x(u,v,w)$

$v(x,y,z)$ $\qquad$ $y(u,v,w)$

$w(x,y,z)$ $\qquad$ $z(u,v,w)$

$$\underbrace{dx\,dy\,dz}_{\text{tiny volume}} = \underbrace{\left|\det\begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix}\right|}_{\substack{\text{volume stretch factor is determinant} \\ \text{of } 3\times 3 \text{ Jacobian matrix.}}} du\,dv\,dw$$





$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$r^2 = x^2 + y^2.$$

HW 4 due Friday.

Review integration in 2D.

Given $f: \mathbb{R}^2 \to \mathbb{R}$ & $D \subseteq \mathbb{R}^2$, define the integral:

$$\iint_D \overbrace{f \, \underbrace{dA}_{\text{tiny piece of area}}}^{\text{tiny volume, or tiny piece of mass, ...}}$$

To compute:

- Choose a coordinate system.
- Parametrize the domain D in this coordinate system.
- Actually compute the integral.

In Cartesian coordinates:

$$dA = dx\,dy \quad \text{“} \partial(x,y) \text{”}$$

Other coordinates ("u,v - substitution")

$$\begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} \Longleftrightarrow \begin{Bmatrix} x(u,v) \\ y(u,v) \end{Bmatrix}$$

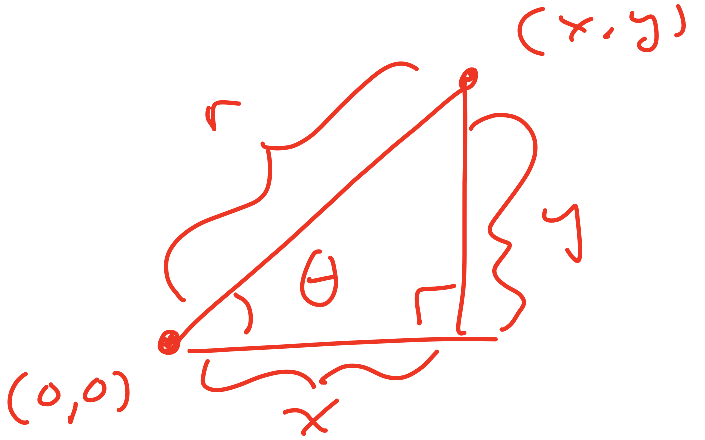
$$dx\,dy = \left| \det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \right| du\,dv$$

More formally:

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$$

Example: Polar Coords

$$\begin{Bmatrix} x = r\cos\theta \\ y = r\sin\theta \end{Bmatrix} \Longleftrightarrow \begin{Bmatrix} r = \sqrt{x^2+y^2} \\ \theta = \arctan(y/x) \end{Bmatrix}$$



$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2 = r^2$

$r = \sqrt{x^2+y^2}$

$y/x = \sin\theta / \cos\theta$

$= \tan\theta$

$$dx\,dy = \left| \det \begin{pmatrix} x_r & x_\theta \\ y_r & y_\theta \end{pmatrix} \right| dr\,d\theta$$

just r

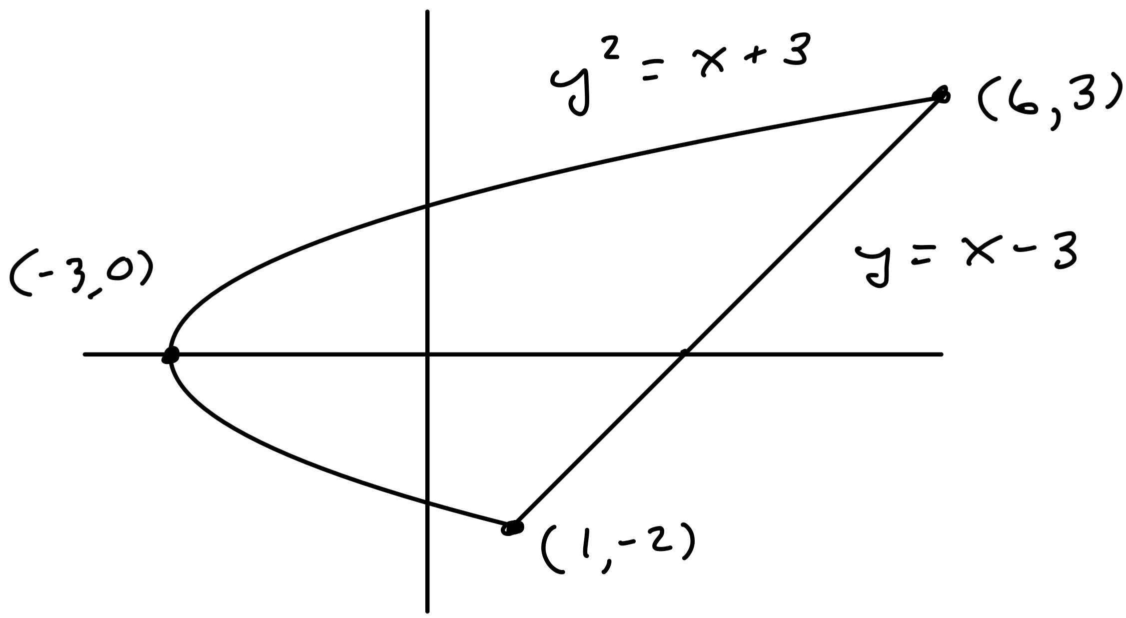
$$dr\,d\theta = \left| \det \begin{pmatrix} r_x & r_y \\ \theta_x & \theta_y \end{pmatrix} \right| dx\,dy$$

HW 4

Sometimes there is no really good coordinate system. Then we probably use Cartesian coords & brute force.

Example: Integrate $\rho(x,y) = 3x^2 + y^2$ over region between parabola $y^2 = x+3$ and line $y = x-3$

Picture:



Interpretation:

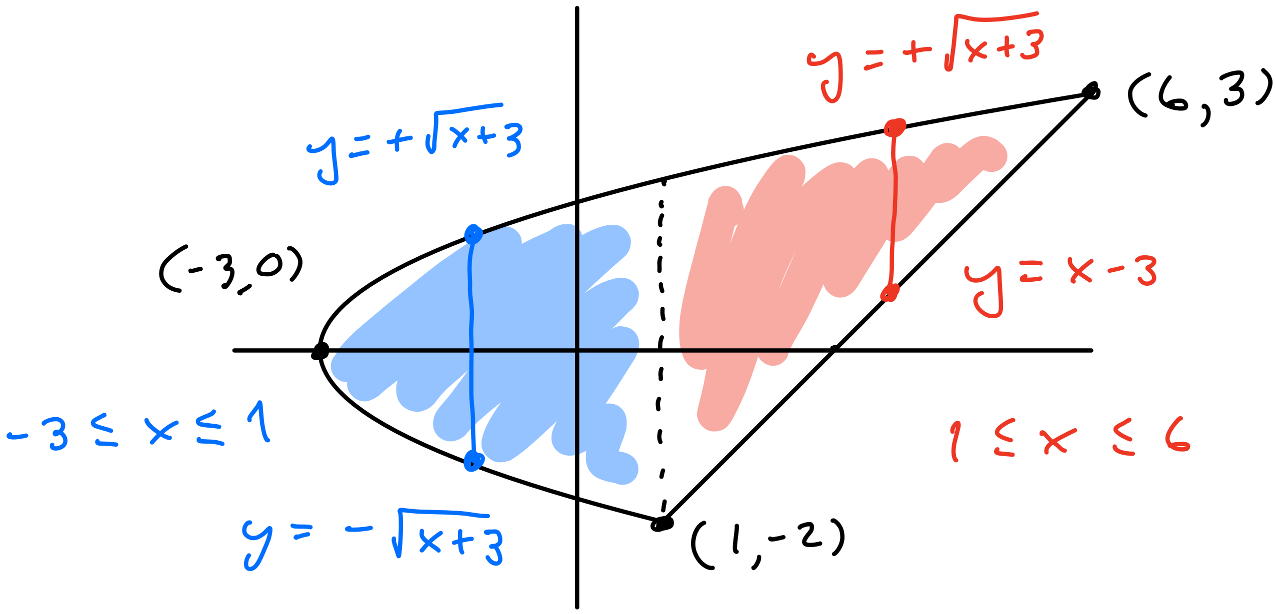
$$\text{mass} = \iint \overbrace{\text{density}\ \underbrace{dA}_{\text{tiny area}}}^{\text{tiny mass}}$$

$$= \iint_D (3x^2 + y^2)\, dx\, dy$$

Parametrize region:

TWO OPTIONS:

- Vertical Slices:



mass = mass of left piece

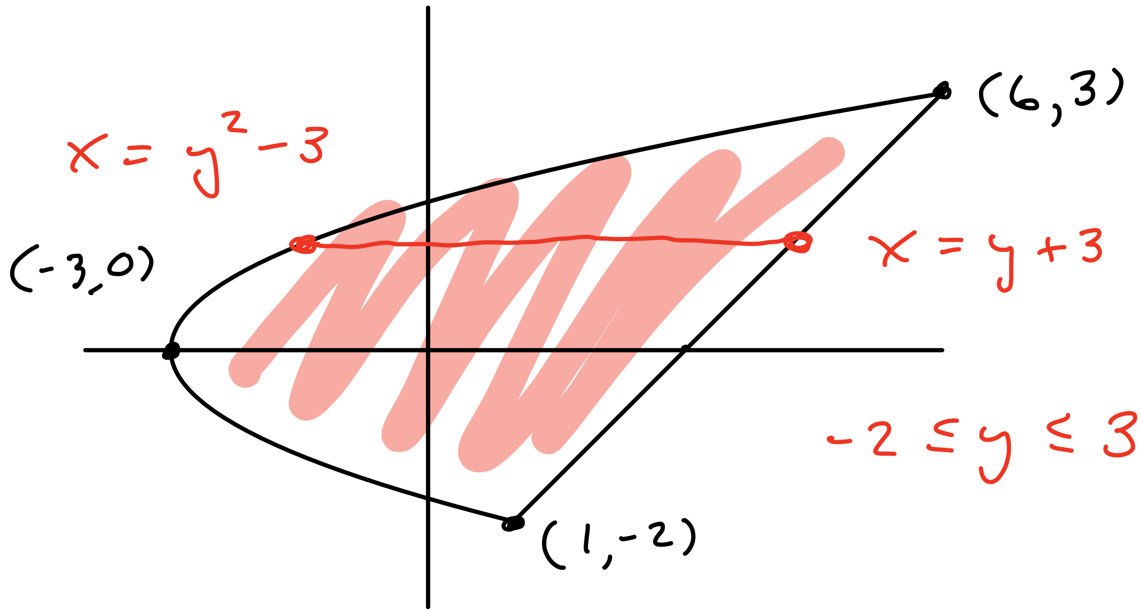
+ mass of right piece

$$= \int_{x=-3}^{1} \left(\int_{y=-\sqrt{x+3}}^{y=+\sqrt{x+3}} (3x^2+y^2)\, dy \right) dx$$

$$+ \int_{x=1}^{6} \left(\int_{y=x-3}^{y=+\sqrt{x+3}} (3x^2+y^2)\, dy \right) dx$$

This looks bad. Skip to next method.

- Horizontal Slices



Two benefits : Only one region
No square roots ‿

$$\text{mass} = \int_{y=-2}^{3} \left(\int_{x=y^2-3}^{y+3} (3x^2+y^2)\,dx \right) dy$$

$$= \int_{y=-2}^{3} \left[\underbrace{3\cdot\frac{x^3}{3} + y^2 x}_{\text{expand}} \right]_{x=y^2-3}^{x=y+3} dy .$$

⋮ SKIP
⋮

$$= \int_{-2}^{3} \overbrace{\left(54 + 27y - 12y^2 + 2y^3 + 8y^4 - y^6\right)}^{\varphi} dy$$

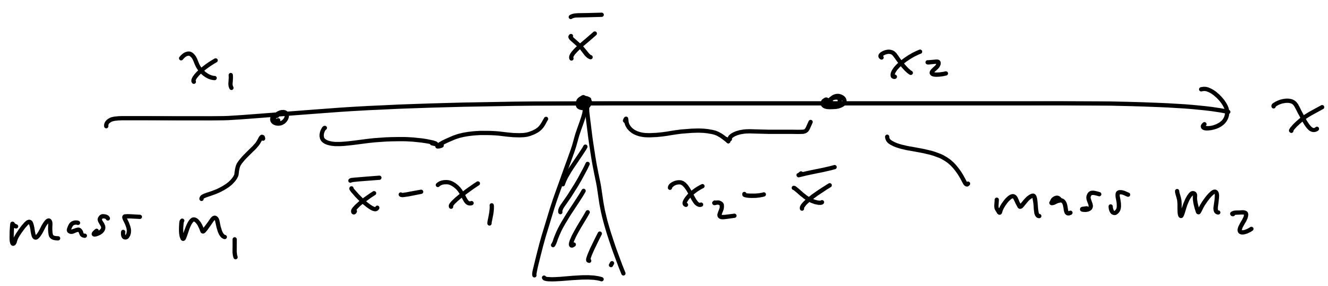
$\vdots$ SKIP (Computer)

$$= \frac{2375}{7} \approx 339 \text{ units of mass.}$$

What is the center of mass?

[This is the point that follows parabolic trajectory when object is thrown in the air.]

Archimedes:



Law of lever says

balance $\iff$ $m_1(\bar{x}-x_1) = m_2(x_2-\bar{x})$

Solve for $\bar{x}$:

$$m_1\bar{x} - m_1x_1 = m_2x_2 - m_2\bar{x}$$

$$(m_1+m_2)\,\bar{x} = m_1x_1 + m_2x_2$$

$$\bar{x} = \frac{m_1x_1 + m_2x_2}{m_1+m_2}$$

Generalize to many point masses:

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + \cdots + m_nx_n}{m_1+m_2+\cdots+m_n}$$

$$= \frac{\sum m_i x_i}{\sum m_i} \quad \text{total mass}$$

For a continuous density $\rho(x)$ on the real line we get

$$\bar{x} = \frac{\int x \rho(x)\, dx}{\int \rho(x)\, dx}$$ total mass

Given $\rho(x,y)$ in 2D we will use the notation

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m} \right)$$

where

$$m = \iint \rho(x,y)\, dx\, dy = \text{total mass}$$

$$M_y = \iint x \rho(x,y)\, dx\, dy$$

= "moment about the y-axis"
(x-coord is distance from y-axis)

$$M_x = \iint y \rho(x,y)\, dx\, dy$$

= "moment about x-axis"

In our example: $\rho(x,y) = 3x^2 + y^2$

Region: $-2 \le y \le 3$

$y^2 - 3 \le x \le y + 3$

$$M_y = \int_{-2}^{3} \left(\int_{y^2-3}^{y+3} x(3x^2+y^2)\,dx \right) dy$$

$= 39875/42$ (computer)

Computer also gives

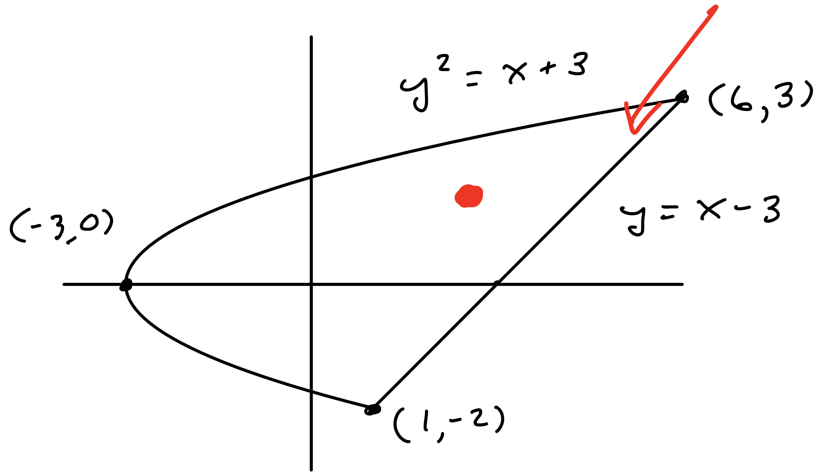
$M_x = 11125/24$

So the center of mass is

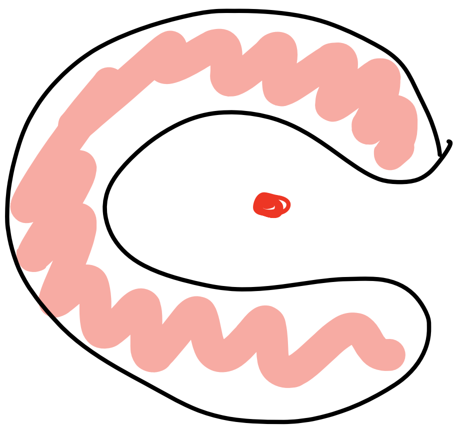
$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m} \right)$$

$= (2.8, 1.4)$

this tail is heavy



Remark: Center of mass need not be inside the region:



The "same formulas" hold in 3D.

Let $\rho(x,y,z)$ = mass per unit volume.

Then total mass is a triple integral:

$$\underset{M}{\text{mass}} = \iiint \underbrace{\rho(x,y,z)\,\overbrace{dx\,dy\,dz}^{\text{tiny piece of volume}}}_{\text{tiny piece of mass}}$$

The center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m} \right)$$

where

$$M_{yz} = \iiint x \rho(x,y,z)\, dx\, dy\, dz$$

= "moment about yz plane"
(x-coord is distance from yz plane)

etc...

HOW to COMPUTE 3D INTEGRAL?
Pretty much the same as 2D.

Example: Volume of a box

$$a_1 \le x \le a_2$$
$$b_1 \le y \le b_2$$
$$c_1 \le z \le c_2$$

$$\text{volume} = \iiint 1 \, dx\, dy\, dz$$

$$= \int_{a_1}^{a_2} dx \int_{b_1}^{b_2} dy \int_{c_1}^{c_2} dz$$

$$= (a_2 - a_1)(b_2 - b_1)(z_2 - z_1)$$

(length)(width)(height)

of course.

[Remark: IF the integrand is "separable" $F(x,y,z) = f(x)g(y)h(z)$ then the integral is a product:

$$\iiint F(x,y,z)\,dx\,dy\,dz$$

$$= \iiint f(x)g(y)h(z)\,dx\,dy\,dz$$

$$= \int f(x)\,dx \int g(y)\,dy \int h(z)\,dz.$$

USEFUL!

e.g. $F(x,y,z) = x^2 e^y \sin(z)$

is separable.

$$F(x,y,z) = e^{xy}\sin(yz)$$

is not separable.]

Center of Mass ?

$$(\bar{x},\bar{y},\bar{z}) = \left(\frac{a_1+a_2}{2}, \frac{b_1+b_2}{2}, \frac{c_1+c_2}{2}\right)$$

SHOULD BE.

(Check)

$$M_{yz} = \int\int\int x\,dx\,dy\,dz$$

$$= \int x\,dx \int dy \int dz$$

$$= \left(\frac{a_2^2-a_1^2}{2}\right)(b_2-b_1)(c_2-c_1)$$

$$\frac{M_{yz}}{m} = \frac{\left(\frac{a_2^2-a_1^2}{2}\right)\cancel{(b_2-b_1)}\cancel{(c_2-c_1)}}{(a_2-a_1)\cancel{(b_2-b_1)}\cancel{(c_2-c_1)}}$$

FACTOR

$$= \frac{(\cancel{a_2 - a_1})(a_2 + a_1)}{2} \cdot \frac{1}{\cancel{a_2 - a_1}}$$

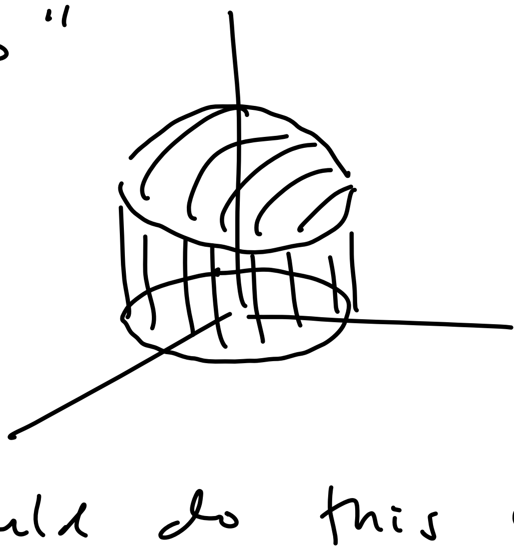
$$= (a_1 + a_2)/2$$

Harder Example:

Compute volume of 3D region E

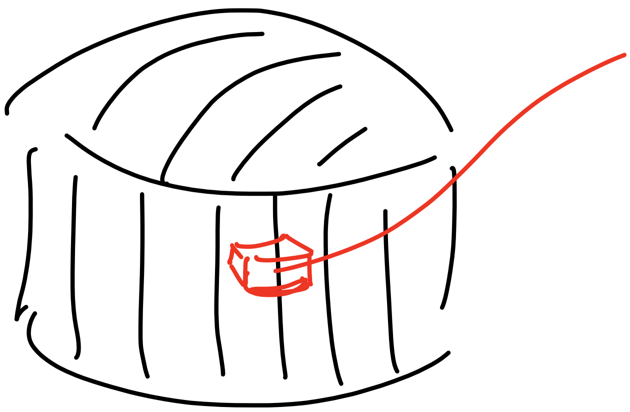
- Above xy plane
- Inside cylinder $x^2 + y^2 \leq 1$
- Inside sphere $x^2 + y^2 + z^2 \leq 4$.

"silo"



We could do this with 2D

integral using polar coords, but we'll use 3D integral for illustration.



tiny piece of volume $dx\,dy\,dz = r\,dr\,d\theta\,dz$

$$\text{Volume} = \iiint 1 \, dx\,dy\,dz$$

$$= \iiint 1 \, r\,dr\,d\theta\,dz.$$

Parametrize E:

$$x^2+y^2 \leq 1 \rightsquigarrow r^2 \leq 1$$

$$\rightsquigarrow r \leq 1.$$

$$x^2+y^2+z^2 \leq 4$$

$$z^2 \leq 4 - x^2 - y^2$$

nice rotational symmetry

$$z^2 \leq 4 - r^2$$

$$0 \le z \le \sqrt{4-r^2}$$

involves r so integrate over z before r.

Also $0 \le \theta \le 2\pi$.

$$\text{volume} = \iiint 1 \, r \, dr \, d\theta \, dz$$

$$= \int_0^{2\pi} d\theta \int_0^1 r \left(\int_0^{\sqrt{4-r^2}} 1 \, dz \right) dr$$

$$= 2\pi \int_0^1 r \sqrt{4-r^2} \, dr$$

$u = 4 - r^2$

$du = -2 r \, dr$ NICE.

$r \, dr = -\frac{1}{2} du$

$$= 2\pi \int_4^3 -\frac{1}{2} \sqrt{u} \, du \qquad \sqrt{u} = u^{1/2}$$

$$= 2\pi \left[-\frac{1}{2} \frac{u^{3/2}}{3/2} \right]_4^3$$

$$= 2\pi \left[-\frac{1}{\cancel{2}} \frac{(3)^{3/2}}{3/\cancel{2}} + \frac{1}{\cancel{2}} \frac{\overset{8}{\cancel{(4)^{3/2}}}}{3/\cancel{2}} \right]$$

NOT SO BAD!

$$= 2\pi \left[\frac{8}{3} - \frac{\cancel{(3)^{3/2}}}{\cancel{3}} \right]$$

$3^{1/2}$

$$= 2\pi \left[\frac{8}{3} - \sqrt{3} \right]$$

HW 4 due tomorrow.

Integration in 3D.

[Chap 5: Integration over 2D regions in $\mathbb{R}^2$ & over 3D regions in $\mathbb{R}^3$.

Chap 6: Integrate along a curve in $\mathbb{R}^2$. Integrate along a curve or surface in $\mathbb{R}^3$.]

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be scalar field

Let $E \subseteq \mathbb{R}^3$ be solid region,

Want to compute

$$\iiint_E \underbrace{f \overbrace{dV}^{\text{tiny piece of volume}}}_{?}$$

f = mass density $\rightsquigarrow$ $f\,dV$ = mass.

f = temperature $\rightsquigarrow$ $f\,dV \approx$ heat energy

[Also: $f(x,y,z)$ = "height above" the xyz-space in xyzw-space.

Then $f\,dV$ = 4D hypervolume.]

To compute:

- Pick coordinate system. } human
- Parametrize region E. } human
- Compute the integral. ← a computer can do this

Cartesian: $dV = dx\,dy\,dz$.

General coordinates:

$$\left\{ \begin{array}{l} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{l} x(u,v,w) \\ y(u,v,w) \\ z(u,v,w) \end{array} \right\}$$

Define the Jacobian determinant

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix}$$

Volume Forms

$$\underbrace{dx\,dy\,dz}_{\text{tiny volume}} = \underbrace{\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|}_{\text{volume stretch factor}} \underbrace{du\,dv\,dw}_{\text{tiny volume}}$$

e.g. Stretch in 3 directions

$$x = au$$
$$y = bv$$
$$z = cw$$

constants a, b, c.

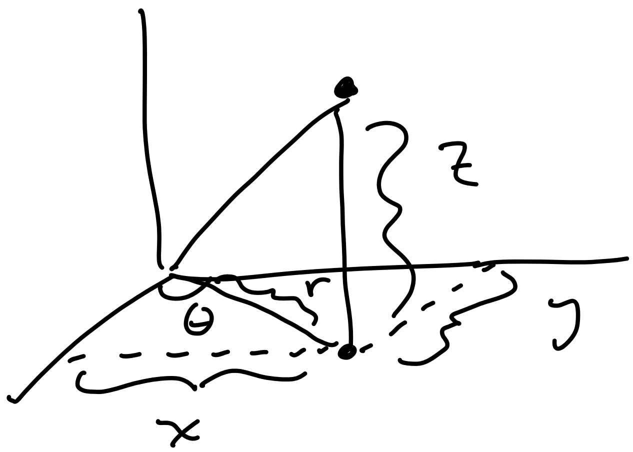
$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

$$= abc.$$

$$dx\,dy\,dz = \underbrace{abc}_{\text{volume stretch factor.}}\ du\,dv\,dw$$

[See Problem 5 on HW 4.]

e.g. Cylindrical Coords.



$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \det\begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

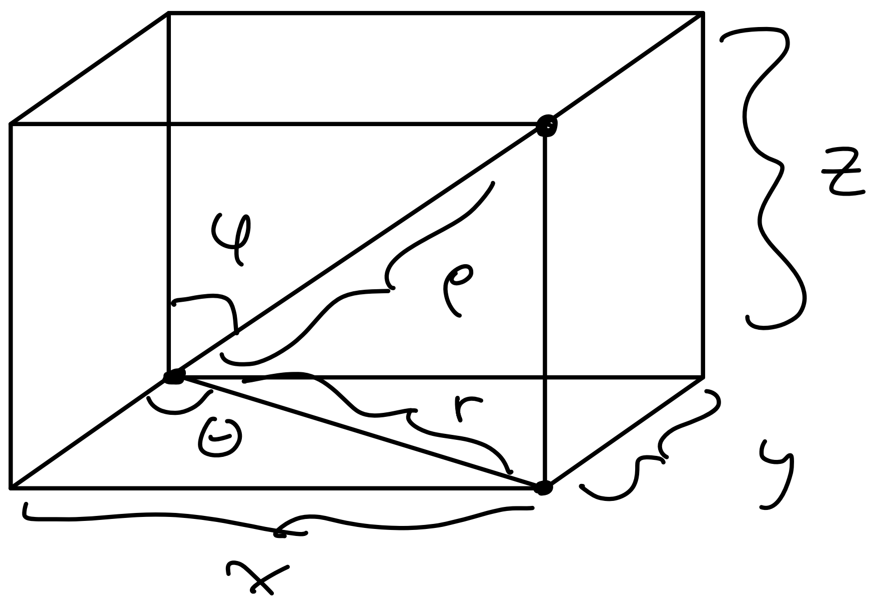
$$= 1 \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$= r$$

$$\underbrace{dx\,dy}\,dz = r\,\underbrace{dr\,d\theta}\,dz$$

Just polar coords with z attached.

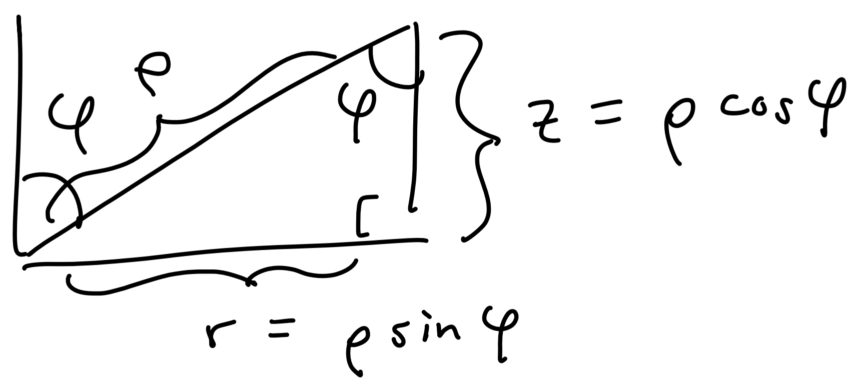
e.g. Spherical Coords



$x = r\cos\theta \qquad r = \rho\sin\varphi$

$y = r\sin\theta \qquad z = \rho\cos\varphi$

$r^2 = x^2 + y^2$



Spherical coords are ρ, θ, φ

$$\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases}$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} =$$

$$\det \begin{pmatrix} \sin\varphi\cos\theta & \sin\varphi\sin\theta & \cos\varphi \\ -\rho\sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & 0 \\ \rho\cos\varphi\cos\theta & \rho\cos\varphi\sin\theta & -\rho\sin\varphi \end{pmatrix}$$

$$= -\rho^2\sin\varphi \quad \text{(via computer).}$$

$$dx\,dy\,dz = |-\rho^2\sin\varphi|\,d\rho\,d\theta\,d\varphi$$

$$= \rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$

That's ugly. Let's make sure that it words. [HW 4.5(a)]

Compute volume of sphere of radius a:

$$x^2 + y^2 + z^2 \leq a$$

$$\text{volume} = \iiint_{\text{sphere}} 1 \, dV$$

Could use Cartesian coords but the parametrization is a mess:

$$-a \leq x \leq a$$
$$-\sqrt{a^2 - x^2} \leq y \leq +\sqrt{a^2 - x^2}$$
$$-\sqrt{a^2 - x^2 - y^2} \leq z \leq +\sqrt{a^2 - x^2 - y^2}$$

Much better to use spherical coords:

$$0 \leq \rho \leq a$$
$$0 \leq \theta \leq 2\pi$$
$$0 \leq \varphi \leq \pi$$

constant !! ☺

$$\text{volume} = \iiint \underbrace{\rho^2 \sin\varphi}_{\text{separable !!}} \, d\rho \, d\theta \, d\varphi$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^{\pi} \sin\varphi \, d\varphi \int_0^{a} \rho^2 \, d\rho$$

$$= 2\pi \left[-\cos\varphi \right]_0^{\pi} \cdot \left[\frac{1}{3}\rho^3 \right]_0^{a}$$

$$= 2\pi \left[\overset{1}{\cancel{-\cos(\pi)}} + \overset{1}{\cos(0)} \right] \cdot \left[\frac{1}{3} a^3 \right]$$

$$= 4\pi \left[\frac{1}{3} a^3 \right]$$

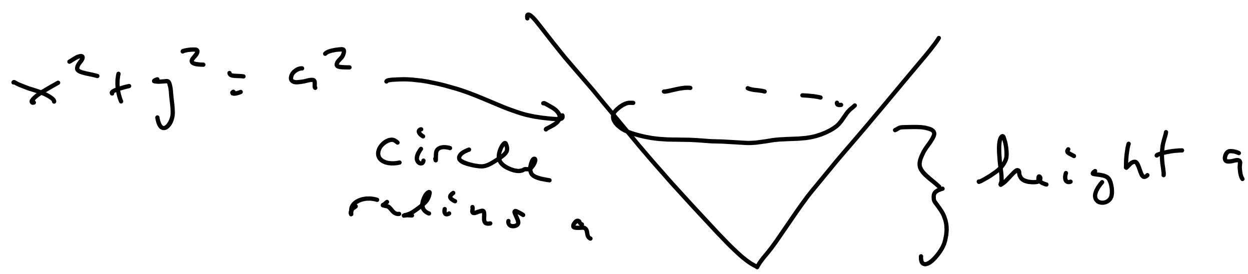
$$= \frac{4}{3}\pi a^3$$

Yes, this is the formula you memorized in 8th grade math.

Harder Example: Find the center of mass of the solid region:

- above the xy-plane
- below the cone $z^2 = x^2 + y^2$
- inside the sphere $x^2 + y^2 + z^2 = 1$.

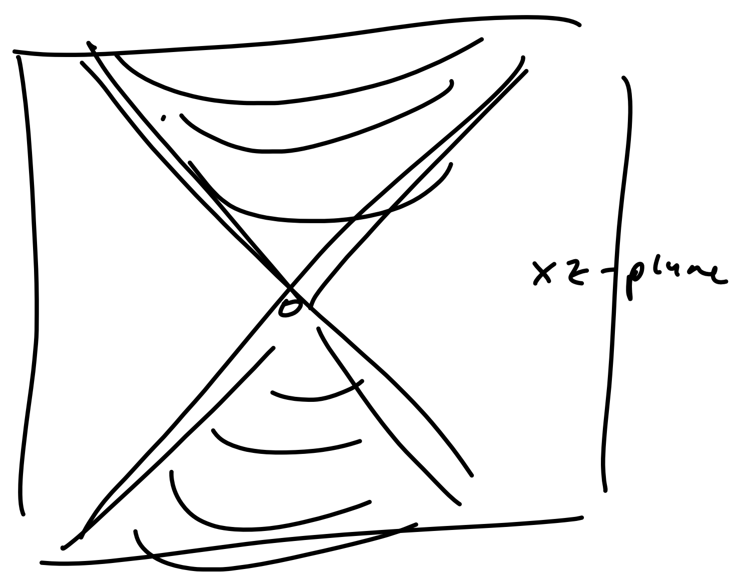
[Why a cone? In the plane $z = a$
The surface is a circle of radius a.



Intersect with plane $y = 0$.

$$x^2 + 0 = z^2$$
$$x^2 - z^2 = 0$$
$$(x - z)(x + z) = 0$$
$$x = \pm z.$$

Two lines



]

Spherical Coordinates:

$$0 \le \rho \le 1$$
$$0 \le \theta \le 2\pi$$
$$\frac{\pi}{4} \le \varphi \le \pi/2 \quad \text{(see picture)}$$

Total Mass:

$$m = \iiint 1 \, dV$$

$$= \iiint \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$= \int_0^{2\pi} d\theta \cdot \int_{\pi/4}^{\pi/2} \sin\varphi \, d\varphi \cdot \int_0^1 \rho^2 \, d\rho$$

$$= 2\pi \left[-\overset{0}{\cancel{\cos\left(\frac{\pi}{2}\right)}} + \overset{\sqrt{2}/2}{\cancel{\cos\left(\frac{\pi}{4}\right)}} \right] \left[\frac{1}{3} \right]$$

$$= \frac{\cancel{2}\pi}{3} \left[\frac{\sqrt{2}}{\cancel{2}} \right] = \frac{\sqrt{2}}{3}\pi .$$

Center of mass:

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\cancel{\frac{M_{yz}}{m}}, \cancel{\frac{M_{xz}}{m}}, \frac{M_{xy}}{m} \right)$$

$$\cancel{M_{yz} = \iiint x \, dV}$$

zero by symmetry

$$\cancel{M_{xz} = \iiint y \, dV}$$

$$M_{xy} = \iiint z \, dV.$$

$$= \iiint z \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$= \iiint \rho \cos\varphi \, \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$= \iiint \rho^3 \frac{1}{2} \sin(2\varphi) \, d\rho \, d\theta \, d\varphi$$

$$= \int_0^{2\pi} d\theta \int_{\pi/4}^{\pi/2} \frac{1}{2} \sin(2\varphi) \, d\varphi \int_0^1 \rho^3 \, d\rho$$

$$= 2\pi \left[-\frac{1}{4} \cos(2\varphi) \right]_{\pi/4}^{\pi/2} \left[\cancel{\frac{1}{4} \rho^4} \right]_0^1$$

$$= \frac{-2\pi}{16}\left[\overset{-1}{\cancel{\cos(\pi)}} - \overset{0}{\cancel{\cos\left(\tfrac{\pi}{2}\right)}}\right]$$

$$= \frac{2\pi}{16} = \frac{\pi}{8}.$$

Finally, the center of mass is:

$$(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{\pi/8}{\sqrt{2}\,\pi/3}\right)$$

$$= \left(0, 0, \frac{3}{\sqrt{2}\cdot 8}\right)$$

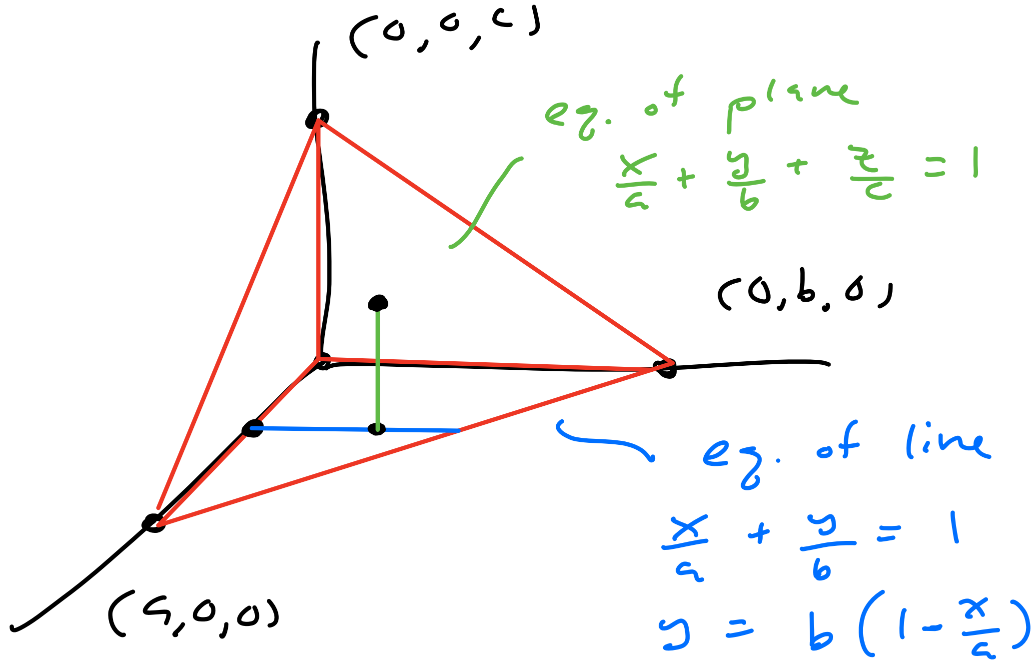
$$= (0, 0, 0.265).$$

Integrating over a Tetrahedron.

Sometimes there is no really good coordinate system.

Consider tetraheron with

vertices $(0,0,0)$, $(a,0,0)$, $(0,b,0)$, $(0,0,c)$:



6 reasonable ways to parametrize this shape.

Fix $\quad 0 \le x \le a$

Then $\quad 0 \le y \le b\left(1 - \frac{x}{a}\right)$

$\quad\quad\quad 0 \le z \le c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$

For any scalar field we have

$$\iiint_{\text{tetrahedron}} F \, dV$$

$$= \underbrace{\int_0^a \underbrace{\left(\int_0^{b(1-\frac{x}{a})} \left(\underbrace{\int_0^{c(1-\frac{x}{a}-\frac{y}{b})} f \, dz}_{\text{formula in } x,y} \right) dy \right)}_{\text{some formula in } x} dx}_{\text{just a number.}}$$