Math 310 **Spring 2024**
Homework 6 **Drew Armstrong**

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1. Various Kinds of First and Second Derivatives in $\mathbb{R}^3$. For any scalar field $f(x,y,z)$ we define a vector field Grad(f) and a scalar field Laplacian(f) by

$$\begin{aligned}\text{Grad}(f) &= \text{“}\nabla f\text{”} = \langle f_x, f_y, f_z\rangle,\\ \text{Laplacian}(f) &= \text{“}\nabla^2 f\text{”} = f_{xx} + f_{yy} + f_{zz}.\end{aligned}$$

and for any vector field $\mathbf{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ we define a vector field Curl($\mathbf{F}$) and a scalar field Div($\mathbf{F}$) by

$$\begin{aligned}\text{Curl}(\mathbf{F}) &= \text{“}\nabla \times \mathbf{F}\text{”} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle,\\ \text{Div}(\mathbf{F}) &= \text{“}\nabla \bullet \mathbf{F}\text{”} = P_x + Q_y + R_z.\end{aligned}$$

(a) For any scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ check that Curl(Grad(f)) $= \langle 0,0,0\rangle$.
(b) For any vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ check that Div(Curl($\mathbf{F}$)) $= 0$.
(c) For any scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ check that Div(Grad(f)) = Laplacian(f).

(a): Let Grad(f) $= \langle P, Q, R\rangle = \langle f_x, f_y, f_z\rangle$. Then

$$\begin{aligned}\text{Curl}(\text{Grad}(f)) &= \text{Curl}(\langle P, Q, R\rangle)\\ &= \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle\\ &= \langle (f_z)_y - (f_y)_z, (f_x)_z - (f_z)_x, (f_y)_x - (f_x)_y\rangle\\ &= \langle f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy}\rangle\\ &= \langle 0,0,0\rangle,\end{aligned}$$

because the mixed partial derivatives of f commute.

(b): Let $\mathbf{F} = \langle P, Q, R\rangle$ and let Curl($\mathbf{F}$) $= \langle U, V, W\rangle$, so that

$$\langle U, V, W\rangle = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle.$$

Then we have

$$\begin{aligned}\text{Div}(\text{Curl}(\mathbf{F})) &= \text{Div}(\langle U, V, W\rangle)\\ &= U_x + V_y + W_z\\ &= (R_y - Q_z)_x + (P_z - R_x)_y + (Q_x - P_y)_z\\ &= R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz}\\ &= P_{zy} - P_{yz} + Q_{xz} - Q_{zx} + R_{yx} - R_{xy}\\ &= 0 + 0 + 0\\ &= 0,\end{aligned}$$

because the mixed partial derivatives of P, Q, R commute.

Remark: Thus two of the most natural concepts of “second derivative” in three dimensions are always zero. There are only two more that we could consider: The divergence of the gradient, and the gradient of the divergence. The divergence of the gradient is called the “Laplacian” and it is the most useful version of “the second derivative” in higher dimensions.

(c): For any scalar function $f(x,y,z)$ we have

$$\text{Div}(\text{Grad}(f)) = \text{Div}(\langle f_x, f_y, f_z\rangle)$$

$$= (f_x)_x + (f_y)_y + (f_z)_z$$
$$= f_{xx} + f_{yy} + f_{zz}.$$

Remark: It is common to think of ∇ as a vector of differential operators

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle = \langle \partial_x, \partial_y, \partial_z \rangle.$$

This is a somewhat fictional concept but the notation is useful.[1] Then we can think of ∇f, $\nabla \times \mathbf{F}$ and $\nabla \bullet \mathbf{F}$ as scalar multiplication, cross product and dot product. Using this notation, parts (a) and (b) become "obvious" because

$$\text{Curl}(\text{Grad}(f)) = \nabla \times (\nabla f) = (\nabla \times \nabla) f = \langle 0,0,0 \rangle f = \langle 0,0,0 \rangle$$

and[2]

$$\text{Div}(\text{Curl}(\mathbf{F})) = \nabla \bullet (\nabla \times \mathbf{F}) = \mathbf{F} \bullet (\nabla \times \nabla) = \mathbf{F} \bullet \langle 0,0,0 \rangle = 0.$$

But are these proofs correct?

In operator notation the Laplacian becomes

$$\text{Div}(\text{Grad}(f)) = \nabla \bullet (\nabla f) = (\nabla \bullet \nabla) f = \text{``}\nabla^2 f\text{''},$$

where

$$\nabla^2 = \nabla \bullet \nabla = (\partial_x)^2 + (\partial_y)^2 + (\partial_z)^2$$

is the "Laplacian differential operator". This operator appears in the fundamental equations of physics. If $f(x, y, z, t)$ is the temperature at the point (x, y, z) at time t then the heat diffusion is governed by the "heat equation":

$$f_t = -\nabla^2 f.$$

If $f(x, y, z, t)$ is the pressure at the point (x, y, z) at time t then the propagation of pressure waves (such as sound) is governed by the "wave equation":

$$f_{tt} = \nabla^2 f.$$

2. Conservative Vector Fields. Consider the vector field

$$\mathbf{F}(x, y, z) = \langle 2x + y, x + z, y \rangle.$$

(a) Check that the curl is zero: $\nabla \times \mathbf{F}(x, y, z) = \langle 0,0,0 \rangle$.

(b) It follows from (a) that there exists a scalar field $f(x, y, z)$ satisfying $\nabla f(x, y, z) = \mathbf{F}(x, y, z)$. Find one example of such a field. [Hint: Integrate $\mathbf{F}$ along an arbitrary path starting at some arbitrary point and ending at the point (x, y, z). For the purpose of this calculation let x, y, z be constant.]

(a): Let $\mathbf{F}(x, y, z) = \langle 2x + y, x + z, y \rangle = \langle P, Q, R \rangle$. Then

$$\begin{aligned}\text{Curl}(\mathbf{F}) &= \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle \\ &= \langle (y)_y - (x+z)_z, (2x+y)_z - (y)_x, (x+z)_x - (2x+y)_y \rangle \\ &= \langle 1-1, 0-0, 1, 1 \rangle \\ &= \langle 0,0,0 \rangle.\end{aligned}$$

This guarantees that there exists a scalar field $f(x, y, z)$ satisfying $\mathbf{F} = \nabla f$.

[1] This "operator calculus" was introduced by Maxwell and other British mathematicians in the late 1800s. It was regarded with suspicion by some mathematicians well into the twentieth century.

[2] It is true for any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ that $\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) = \mathbf{w} \bullet (\mathbf{u} \times \mathbf{v})$.

(b): To find such a field we should integrate $\mathbf{F}$ along a path $\mathbf{r}(t)$ ending at the point (x, y, z). (For the purpose of this calculation we let x, y, z be constant.) The simplest such path I can think of is $\mathbf{r}(t) = (tx, ty, tz)$ with $0 \leq t \leq 1$. Thus we get

$$\begin{aligned}
f(x, y, z) &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt \\
&= \int_0^1 \mathbf{F}(tx, ty, tz) \bullet \langle x, y, z\rangle\, dt \\
&= \int_0^1 \langle 2tx + ty, tx + tz, ty\rangle \bullet \langle x, y, z\rangle\, dt \\
&= \int_0^1 [(2tx + ty)x + (tx + tz)y + (ty)z]\, dt \\
&= \int_0^2 [2x^2 + yx + xy + zy + yz]t\, dt \\
&= (2x^2 + 2xy + 2yz) \int_0^1 t\, dt \\
&= (2x^2 + 2xy + 2yz)\, \frac{1}{2}t^2 \Big|_0^1 \\
&= x^2 + xy + yz.
\end{aligned}$$

It is easy to check that this answer is correct:

$$\nabla(x^2 + xy + yz) = \langle 2x + y, x + z, y\rangle = \mathbf{F}.$$

Remark: If $\mathbf{F}$ were *not conservative*, i.e., if $\text{Curl}(\mathbf{F}) \neq \langle 0, 0, 0\rangle$, we could still try to compute $f(x, y, z)$ using the above method, but the final answer would *not* satisfy $\nabla f = \mathbf{F}$ because a non-conservative field can not be expressed in the form ∇f.

3. Green's Theorem on a Rectangle. Consider the vector field $\mathbf{F}(x, y) = \langle y^2, x^2\rangle$.

(a) Compute the scalar curl of $\mathbf{F}$.
(b) Integrate the scalar curl of $\mathbf{F}$ over the rectangle with $0 \leq x \leq 2$ and $0 \leq y \leq 1$.
(c) Let C_1, C_2, C_3, C_4 be the four sides of the rectangle, oriented counterclockwise. Integrate $\mathbf{F}$ along each of these curves and add the results. Check that your answers to (a) and (b) are the same. [Hint: You can parametrize the four sides by

$$\begin{aligned}
\mathbf{r}_1(t) &= (0, 0) + t(2, 0), \\
\mathbf{r}_2(t) &= (2, 0) + t(0, 1), \\
\mathbf{r}_3(t) &= (2, 1) + t(-2, 0), \\
\mathbf{r}_4(t) &= (0, 1) + t(0, -1),
\end{aligned}$$

each with $0 \leq t \leq 1$.]

(a): Let $\mathbf{F}(x, y) = \langle y^2, x^2\rangle = \langle P, Q\rangle$, so that

$$\text{Curl}(\mathbf{F}) = Q_x - P_y = 2x - 2y.$$

Note that $\text{Curl}(\mathbf{F}) \neq 0$ so this field is *not conservative*.

(b): We integrate Curl($\mathbf{F}$) $= 2x - 2y$ over the rectangle D with $0 \le x \le 2$ and $0 \le y \le 1$:

$$\begin{aligned}
\iint_D \mathrm{Curl}(\mathbf{F})\, dA &= \iint_D (2x - 2y) dx dy \\
&= \int_0^1 \left(\int_0^2 (2x - 2y)\, dx \right) dy \\
&= \int_0^1 \left[x^2 - 2yx\right]_0^2 dy \\
&= \int_0^1 \left[x^2 - 2yx\right]_0^2 dy \\
&= \int_0^1 (4 - 4y)\, dy \\
&= \left[4y - 2y^2\right]_0^1 \\
&= 2.
\end{aligned}$$

(c): The boundary of the rectangle is $\partial D = C_1 + C_2 + C_3 + C_4$, so that

$$\int_{\partial D} \mathbf{F} \bullet \mathbf{T}\, dx = \int_{C_1} \mathbf{F} \bullet \mathbf{T}\, ds + \int_{C_2} \mathbf{F} \bullet \mathbf{T}\, ds + \int_{C_3} \mathbf{F} \bullet \mathbf{T}\, ds + \int_{C_4} \mathbf{F} \bullet \mathbf{T}\, ds.$$

Thus we need to compute the four integrals and them add them up. According to Green's Theorem we must get the same answer as in part (a), i.e., 2. For $\mathbf{r}_1(t) = (2t, 0)$ we have

$$\begin{aligned}
\int_{C_1} \mathbf{F} \bullet \mathbf{T}\, ds &= \int_0^1 \mathbf{F}(\mathbf{r}_1(t)) \bullet \mathbf{r}_1'(t)\, dt \\
&= \int_0^1 \mathbf{F}(2t, 0) \bullet \langle 2, 0 \rangle\, dt \\
&= \int_0^1 \langle 0, 4t^2 \rangle \bullet \langle 2, 0 \rangle\, dt \\
&= \int_0^1 0\, dt \\
&= 0.
\end{aligned}$$

Then for $\mathbf{r}_2(t) = (2, t)$ we have

$$\begin{aligned}
\int_{C_2} \mathbf{F} \bullet \mathbf{T}\, ds &= \int_0^1 \mathbf{F}(\mathbf{r}_2(t)) \bullet \mathbf{r}_2'(t)\, dt \\
&= \int_0^1 \mathbf{F}(2, t) \bullet \langle 0, 1 \rangle\, dt \\
&= \int_0^1 \langle t^2, 4 \rangle \bullet \langle 0, 1 \rangle\, dt \\
&= \int_0^1 4\, dt \\
&= 4.
\end{aligned}$$

For $\mathbf{r}_3(t) = (2 - 2t, 1)$ we have

$$\int_{C_3} \mathbf{F} \bullet \mathbf{T}\, ds = \int_0^1 \mathbf{F}(\mathbf{r}_3(t)) \bullet \mathbf{r}_3'(t)\, dt$$

$$
\begin{aligned}
&= \int_0^1 \mathbf{F}(2-2t,1) \bullet \langle -2t, 0\rangle\, dt \\
&= \int_0^1 \langle 1, (2-2t)^2\rangle \bullet \langle -2, 0\rangle\, dt \\
&= \int_0^1 -2\, dt \\
&= -2.
\end{aligned}
$$

And for $\mathbf{r}_4(t) = (0, 1-t)$ we have

$$
\begin{aligned}
\int_{C_4} \mathbf{F} \bullet \mathbf{T}\, ds &= \int_0^1 \mathbf{F}(\mathbf{r}_4(t)) \bullet \mathbf{r}_4'(t)\, dt \\
&= \int_0^1 \mathbf{F}(0, 1-t) \bullet \langle 0, -1\rangle\, dt \\
&= \int_0^1 \langle (1-t)^2, 0\rangle \bullet \langle 0, -1\rangle\, dt \\
&= \int_0^1 0\, dt \\
&= 0.
\end{aligned}
$$

Note that the four integrals add up to 2 as required by Green's Theorem:

$$
\int_{\partial D} \mathbf{F} = \int_{C_1} \mathbf{F} + \int_{C_2} \mathbf{F} + \int_{C_3} \mathbf{F} + \int_{C_4} \mathbf{F} = 0 + 4 - 2 + 0 = 2 = \iint_D \mathrm{Curl}(\mathbf{F}).
$$

4. Stokes' Theorem on a Pringle. Consider the constant vector field $\mathbf{F}(x,y,z) = \langle -y, x, 1\rangle$ and the pringle-shaped surface D defined by

$$
\mathbf{r}(u,v) = \langle u\cos v, u\sin v, u^2 \cos v \sin v\rangle,
$$

with $0 \leq u \leq 1$ and $0 \leq v \leq 2\pi$.

(a) Compute the curl $\nabla \times \mathbf{F}(x,y,z)$.

(b) Compute the flux of the curl $\nabla \times \mathbf{F}$ across the pringle:

$$
\begin{aligned}
\iint_D (\nabla \times \mathbf{F}) \bullet \mathbf{N}\, dA &= \iint (\nabla \times \mathbf{F})(\mathbf{r}(u,v)) \bullet \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} \|\mathbf{r}_u \times \mathbf{r}_v\|\, dudv \\
&= \iint (\nabla \times \mathbf{F})(\mathbf{r}(u,v)) \bullet (\mathbf{r}_u \times \mathbf{r}_v)\, dudv.
\end{aligned}
$$

[Hint: From the previous homework we have $\mathbf{r}_u \times \mathbf{r}_v = \langle -u^2 \sin v, -u^2 \cos v, u\rangle$.]

(c) Let $C = \partial D$ be the boundary curve of the pringle. Compute the circulation of $\mathbf{F}$ around C. Check that your answers to (b) and (c) are the same. [Hint: If $\mathbf{r}(t)$ is a parametrization of C then the circulation is defined by

$$
\begin{aligned}
\int_C \mathbf{F} \bullet \mathbf{T}\, ds &= \int \mathbf{F}(\mathbf{r}(t)) \bullet \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \|\mathbf{r}'(t)\|\, dt \\
&= \int \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt.
\end{aligned}
$$

You can take $\mathbf{r}(t) = \langle \cos t, \sin t, \cos t \sin t\rangle$ with $0 \leq t \leq 2\pi$. At the very end you will need the trig identity $2\cos^2 t = 1 + \cos(2t)$.]

(a): Let $\mathbf{F}(x,y,z) = \langle -y, x, 1\rangle = \langle P, Q, R\rangle$. Then the curl is

$$\begin{aligned}\nabla \times \mathbf{F} &= \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle \\ &= \langle (1)_y - (x)_z, (-y)_z - (1)_x, (x)_x - (-y)_y\rangle \\ &= \langle 0-0, 0-0, 1-(-1)\rangle \\ &= \langle 0,0,2\rangle.\end{aligned}$$

The curl is constant and non-zero.

(b): From the previous homework we have $\mathbf{r}_u \times \mathbf{r}_v = \langle -u^2 \sin v, -u^2\cos v, u\rangle$, so the flux of $\mathbf{F}$ across the surface $\mathbf{r}(u,v)$ with $0 \le u \le 1$ and $0 \le v \le 2\pi$ is

$$\begin{aligned}\iint_D (\nabla\times\mathbf{F}) \bullet \mathbf{N} dA &= \iint (\nabla \times \mathbf{F})(\mathbf{r}(u,v)) \bullet (\mathbf{r}_u \times \mathbf{r}_v)\, dudv \\ &= \iint \langle 0,0,2\rangle \bullet \langle -u^2\sin v, -u^2\cos v, u\rangle\, dudv \\ &= \iint 2u\, dudv \\ &= 2\int_0^{2\pi} dv \int_0^1 u\, du \\ &= 2(2\pi)(1/2) \\ &= 2\pi.\end{aligned}$$

(c): The boundary curve of D can be parametrized by $\mathbf{r}(t) = \langle \cos t, \sin t, \cos t \sin t\rangle$ with $0 \le t \le 2\pi$. The velocity of this parametrization is

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, \cos^2 t - \sin^2 t\rangle.$$

Hence the circulation of $\mathbf{F}$ around the curve is

$$\begin{aligned}\int \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt &= \int \mathbf{F}(\cos t, \sin t, \cos t \sin t) \bullet \langle -\sin t, \cos t, \cos^2 t - \sin^2 t\rangle\, dt \\ &= \int \langle -\sin t, \cos t, 1\rangle \bullet \langle -\sin t, \cos t, \cos^2 t - \sin^2\rangle\, dt \\ &= \int [(-\sin t)(-\sin t) + (\cos t)(\cos t) + (1)(\cos^2 t - \sin^2 t)]\, dt \\ &= \int [\sin^2 t + \cos^2 t + \cos^2 t - \sin^2 t]\, dt \\ &= \int 2\cos^2 t\, dt \\ &= \int [1 + \cos(2t)]\, dt \\ &= \left[t + \frac{1}{2}\sin(2t)\right]_0^{2\pi} \\ &= (2\pi + 0) - (0+0) \\ &= 2\pi,\end{aligned}$$

which is equal to our answer from part (b), as required by Stokes' Theorem.