1. Integrating a Scalar Along a Parabola. Let C be the portion of the parabola $y = x^2$ between points $(0,0)$ and $(1,1)$, and consider the scalar function $f(x,y) = x$. Compute the integral of f along C:

$$\int_C f\,ds.$$

[Hint: You must choose a parametrization of C. I recommend $\mathbf{r}(t) = (t, t^2)$ with $0 \leq t \leq 1$. The resulting integral may be computed by hand using substitution.]

2. Projection. Let $\mathbf{F}$ and $\mathbf{u}$ be any vectors with $\|\mathbf{u}\| = 1$ and let $\mathbf{p}$ be the *component of* $\mathbf{F}$ *in the direction of the unit vector* $\mathbf{u}$.

(a) Since $\mathbf{p}$ is parallel to $\mathbf{u}$ we know that $\mathbf{p} = t\mathbf{u}$ for some scalar t. Use the fact that the vector $\mathbf{p} - \mathbf{F}$ is perpendicular to $\mathbf{u}$ to prove that $t = \mathbf{F} \bullet \mathbf{u}$.
(b) Draw a picture showing the vectors $\mathbf{F}$, $\mathbf{u}$ and $\mathbf{p}$.

3. Integrating Vector Fields Along Different Curves. By definition, the integral of a vector field $\mathbf{F}$ along a parametrized curve $\mathbf{r}(t)$ is

$$\int \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\,dt.$$

Consider the two fields $\mathbf{F}(x,y) = \langle 2xy, x^2 \rangle$, $\mathbf{G}(x,y) = \langle 2y, x^2 \rangle$, and the two different curves $\mathbf{r}(t) = (t,t)$ and $\mathbf{s}(t) = (t, t^2)$ between the points $(0,0)$ and $(1,1)$.

(a) Integrate $\mathbf{F}$ along $\mathbf{r}(t)$ and $\mathbf{s}(t)$. Observe that you get the same answer.
(b) Integrate $\mathbf{G}$ along $\mathbf{r}(t)$ and $\mathbf{s}(t)$. Observe that you don't get the same answer.

4. Area of a Pringle. Let D be the surface in $\mathbb{R}^3$ defined by $z = xy$ and $x^2 + y^2 \leq 1$, which looks like a pringle chip. We can parametrize this region by

$$\mathbf{r}(u,v) = \langle u\cos v, u\sin v, (u\cos v)(u\sin v) \rangle$$

with $0 \leq u \leq 1$ and $0 \leq v \leq 2\pi$.

(a) Compute the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$. [Hint: Use the identity $\sin(2v) = 2\sin v\cos v$.]
(b) Compute the cross product $\mathbf{r}_u \times \mathbf{r}_v$.
(c) Compute the length $\|\mathbf{r}_u \times \mathbf{r}_v\|$ and simplify as much as possible. [Hint: The answer is $\|\mathbf{r}_u \times \mathbf{r}_v\| = u\sqrt{u^2+1}$.]
(d) Use your answer from part (c) to compute the area of the pringle:

$$\text{Area}(D) = \iint_D 1\,\|\mathbf{r}_u \times \mathbf{r}_v\|\,dudv.$$

5. Proof of Conservation of Energy. A *conservative force field* $\mathbf{F}$ has the form $\mathbf{F} = -\nabla f$ for some *scalar potential* f. Suppose that a particle of mass m travels along a trajectory $\mathbf{r}(t)$. Newton says that the force $\mathbf{F}$ acting on the particle satisties

$$\mathbf{F}(\mathbf{r}(t)) = m\mathbf{r}''(t).$$

The *kinetic energy* of the particle at time t is $\mathrm{KE}(t) = \frac{1}{2}m\|\mathbf{r}'(t)\|^2$, the *potential energy* at time t is $\mathrm{PE}(t) = f(\mathbf{r}(t))$, and the *total mechanical energy* is $E(t) = \mathrm{KE}(t) + \mathrm{PE}(t)$. Use the chain rule and product rule for derivatives to show that

$$E'(t) = 0.$$

[Hint: Write $\|\mathbf{r}'(t)\|^2 = \mathbf{r}'(t) \bullet \mathbf{r}'(t)$.]

6. Application of Conservation of Energy. Choose a coordinate system in $\mathbb{R}^3$ with the sun at position $(0, 0, 0)$. Suppose that the sun has mass M. If $\mathbf{F}(x, y, z)$ is the gravitational force exerted by the sun on a spaceship of mass m at position (x, y, z), Newton tells us that $\mathbf{F}(x, y, z) = -\nabla f(x, y, z)$, where[1]

$$f(x, y, z) = -1 \cdot \frac{GMm}{\sqrt{x^2 + y^2 + z^2}}$$

is called the *gravitational potential*. At a certain time, the spaceship has speed s_0 and distance d_0 from the origin. At a later time the spaceship has speed s_1 and distance d_1 from the origin. Use conservation of energy to compute s_1 in terms of s_0, d_0 and d_1. (Assume that no other forces are acting on the spaceship.)

[1] G is a constant of nature called the *gravitational constant*.