This week : Chapter 6

Next week :

- Mon no class
- ~~Quiz 5~~ HW 5 due on Tuesday
- Wed ~~"bonus lecture"~~ Quiz 5
- Thurs & Fri no class.
- Final Project due Fri.

Chap 6 :

- Integrating over curves & surfaces
- vector field definitions (divergence & curl)
- "Fundamental Theorems"

Calc I & II : $\int F' = f$

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

New : $\int \nabla f = f$

$$\iint_{\text{surface}} \nabla \times F = \int_{\text{boundary curve}} F$$

Stokes / Green's Theorem

$$\iiint_{\text{solid}} \nabla \cdot F = \iint_{\text{boundary surface}} F$$

Divergence Theorem

Don't have time to discuss these in detail.

Integrate along a curve. WHY?

Consider a wire C in 3D.

$\rho(x,y,z)$ = mass density of the wire at point (x,y,z)

= mass / unit length.

The mass of the wire:

$$\text{mass} = \int_C \overbrace{\rho \underbrace{ds}_{\text{tiny piece of length}}}^{\text{tiny piece of mass}}$$

To compute this we need to parametrize the curve. Say

$$C : \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

for $a \le t \le b$.

[Here "t" is not really time; it's just a name for parameter.]

Then we define:

$$\int_C \rho \, ds = \int_a^b \rho(\vec{r}(t)) \underbrace{\|\vec{r}\,'(t)\| \, dt}_{\text{tiny piece of length}}$$

tiny mass at point $\vec{r}(t)$

Special Case: $\rho = 1$ then

arc length = mass

$$= \int_a^b 1 \, \|\vec{r}\,'(t)\| \, dt$$

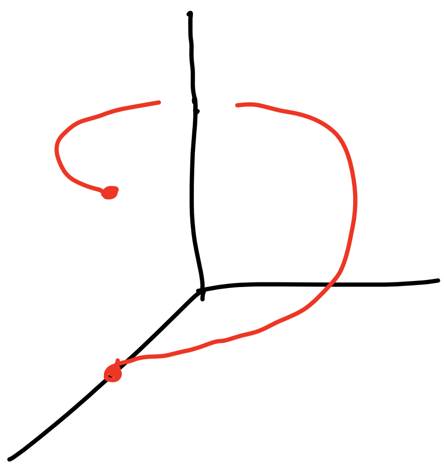
as we already know!

Example: Consider helix

$$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$$

$$0 \le t \le 2\pi.$$

Suppose density $\rho(x,y,z) = x^2 + y^2 + z$.



mass of wire $= \displaystyle\int_C \rho \, ds$

$$\vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{\underbrace{\cancel{\sin^2 t + \cos^2 t}}_{1} + 1^2} = \sqrt{2}$$

Density at point $\vec{r}(t)$:

$$\rho(\vec{r}(t)) = \overset{x^2}{(\cos t)^2} + \overset{y^2}{(\sin t)^2} + \overset{z}{(t)}$$

$$= 1 + t$$ nice

$$\text{mass} = \int_0^{2\pi} \rho(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt$$

$$= \int_0^{2\pi} (1+t)\sqrt{2}\,dt$$

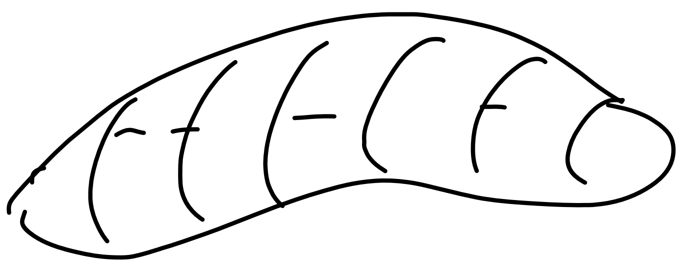
$$= \sqrt{2}\left[t + \frac{1}{2}t^2\right]_0^{2\pi}$$

$$= \sqrt{2}\left[(2\pi) + \frac{1}{2}(2\pi)^2\right].$$

DONE.

Integrating over a surface in $\mathbb{R}^3$.

WHY? Let D be 2D region living in 3D



$\rho(x,y,z) = $ Mass density / area

$$\text{mass of } D = \iint_D \overbrace{\rho \underbrace{dA}_{\text{tiny piece of area}}}^{\text{tiny piece of mass.}}$$

Special case:

$$\text{surface area of } D = \iint_D 1 \, dA .$$

HOW TO COMPUTE ?

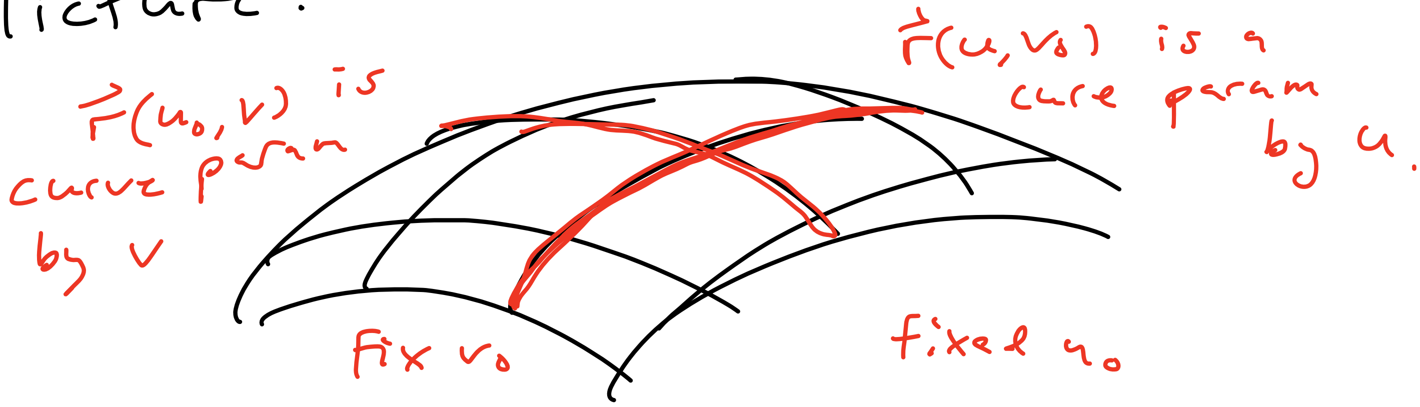
Need to parametrize the surface.

Can think of this as a function

$$\vec{r} : \mathbb{R}^2 \to \mathbb{R}^3$$

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$

Picture:



IDEA :

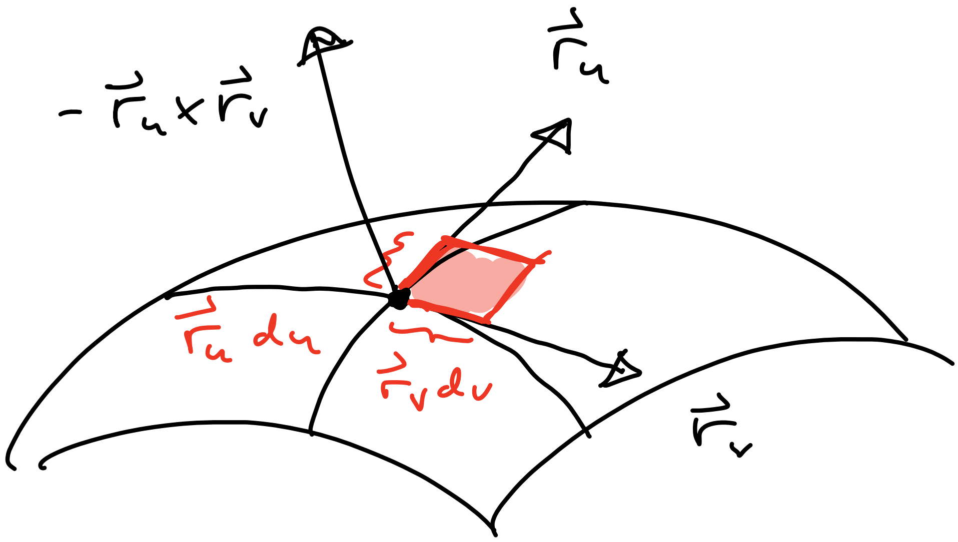
$$\text{mass} = \iint_D \rho(\vec{r}(u,v))\, du\, dv$$

NO! Need some kind of "Jacobian stretch factor"

Different ways to explain this.

Here's the most intuitive:

TWO "VELOCITY VECTORS"



Area of tiny parallelogram

scalars

$$dA = \|(\vec{r}_u\, du) \times (\vec{r}_v\, dv)\|$$

$$= \underbrace{\|\vec{r}_u \times \vec{r}_v\|}\, du\, dv$$

"Jacobian stretch factor"

[More Highbrow:

$$J_{\vec{F}} = \begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix}$$

strech factor

$$= \sqrt{\det\left(J_{\vec{F}}^{T} J_{\vec{F}} \right)} \; . \quad \Big]$$

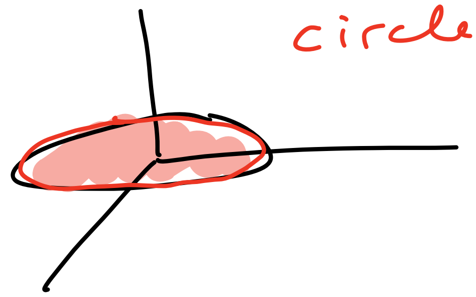
$$\text{mass} = \iint_D \rho(\vec{r}(u,v)) \, \|\vec{r}_u \times \vec{r}_v\| \, du\, dv .$$

$$\text{surface area} = \iint_D 1 \, \|\vec{r}_u \times \vec{r}_v\| \, du\, dv .$$

Test: Area of a Circle in xy-plane

$$\vec{r}(u,v) = \langle u\cos v, \; u \sin v, \; 0 \rangle$$

$$0 \le u \le a \quad \& \quad 0 \le v \le 2\pi$$



$$\vec{r}_u = \langle \cos v, \sin v, 0 \rangle$$

$$\vec{r}_v = \langle -u\sin v, u\cos v, 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle 0, 0, u\cos^2 v + u\sin^2 v \rangle$$

$$= \langle 0, 0, u \rangle$$

$$\| \vec{r}_u \times \vec{r}_v \| = u \qquad (u \geq 0)$$

$$\text{surface area} = \iint \| \vec{r}_u \times \vec{r}_v \| \, du \, dv$$

$$= \iint u \, du \, dv$$

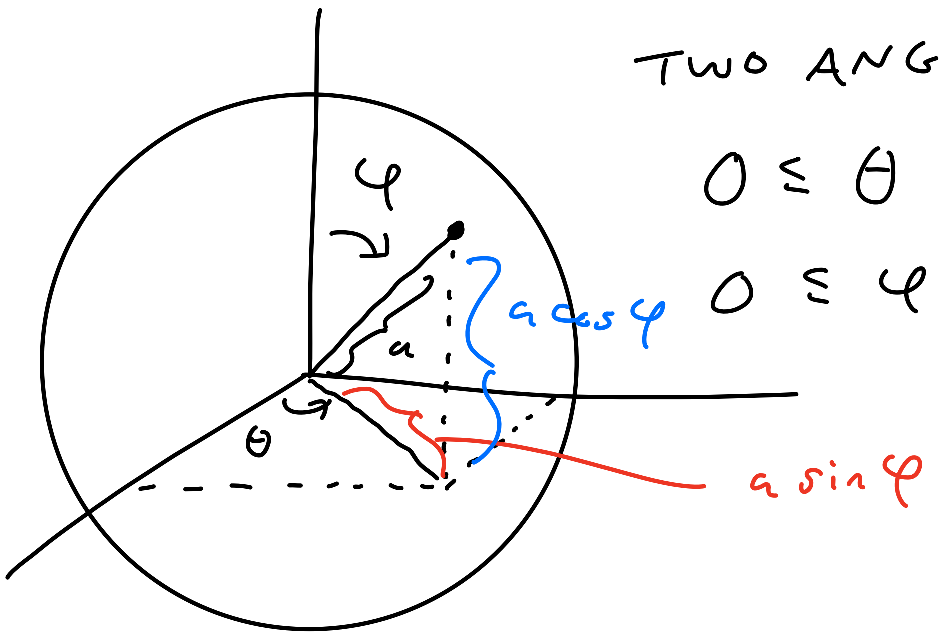
$$= \int_0^{2\pi} dv \int_0^a u \, du$$

$$= 2\pi \left[\frac{1}{2} u^2 \right]_0^a$$

$$= \pi a^2 \quad \checkmark \quad \text{area of circle.}$$

More interesting: Surface area of a sphere of radius a.

Parametrization?



TWO ANGLES:

$$0 \le \theta \le 2\pi$$

$$0 \le \varphi \le \pi$$

Cartesian?

$$x = a \sin\varphi \cos\theta$$

$$y = a \sin\varphi \sin\theta$$

$$z = a \cos\varphi$$

Parametrization:

$$\vec{r}(\theta,\varphi) = \langle x(\theta,\varphi), y(\theta,\varphi), z(\theta,\varphi) \rangle$$

$$= \langle a\cos\theta\sin\varphi,\ a\sin\theta\sin\varphi,\ a\cos\varphi \rangle$$

$$\vec{r}_\theta = \langle -a\sin\theta\sin\varphi,\ a\cos\theta\sin\varphi,\ 0 \rangle$$

$$\vec{r}_\varphi = \langle a\cos\theta\cos\varphi,\ a\sin\theta\cos\varphi,\ -a\sin\varphi \rangle$$

$$\vec{r}_\theta \times \vec{r}_\varphi =$$

$$\langle -a^2\cos\theta\sin^2\varphi,\ a^2\sin\theta\sin^2\varphi,$$

$$-a^2\sin^2\theta\sin\varphi\cos\varphi - a^2\cos^2\theta\sin\varphi\cos\varphi \rangle$$

$$-a^2\sin\varphi\cos\varphi$$

$$\|\vec{r}_\theta \times \vec{r}_\varphi\| = \sqrt{a^4\cancel{\cos^2\theta}\sin^4\varphi + a^4\cancel{\sin^2\theta}\sin^4\varphi + a^4\sin^2\varphi\cos^2\varphi}$$

$$= \ldots = a^2\sin\varphi$$

$\frown$

$$\text{surface area} = \iint_D \|\vec{r}_\theta \times \vec{r}_\varphi\| \, d\theta \, d\varphi$$

$$= \iint a^2 \sin\varphi \, d\theta \, d\varphi$$

$$= a^2 \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi \, d\varphi$$

$$= 2\pi a^2 \left[-\underbrace{\cancel{\cos(\pi)}}_{1} + \underbrace{\cancel{\cos}(0)}_{+1} \right]$$

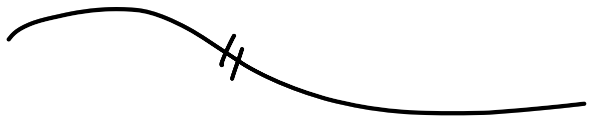
$$= 4\pi a^2 \quad \checkmark$$

surface area of a sphere of radius a.

Next week:

- No class on Mon
- HW5 due Tues
- Quiz 5 on Wed

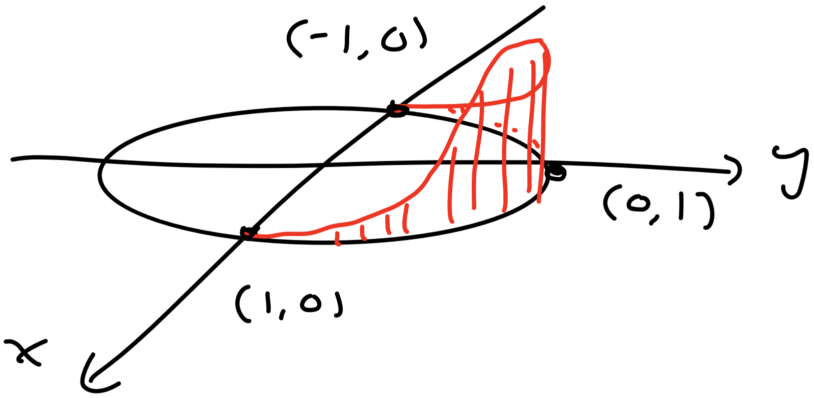
} moved one class later



Last Time: Integrating a scalar field f over a curve in $\mathbb{R}^2, \mathbb{R}^3$ or a surface in $\mathbb{R}^3$.

$$\int_{\text{curve}} \overbrace{f\, \underbrace{ds}_{\text{tiny piece of arc length on curve}}}^{\text{tiny piece of mass, or area, ...}}$$

e.g. Find the area of vertical wall above circle $x^2+y^2=1$ in x,y-plane & below parabolic surface $z=x^2$, with $y \geq 0$. Picture:



Area of wall $= \int$ area of skinny rectangles

$$= \int \underset{\text{height}}{x^2} \, \underset{\text{length of base.}}{ds}$$

To compute this we parametrize the base curve:

$$\vec{r}(t) = \langle \cos t, \sin t \rangle$$
$$t = 0 \text{ to } \pi.$$

According to definition:

$$\int f \, ds = \int f(\vec{r}(t)) \, \|\vec{r}\,'(t)\| \, dt$$

In our case:

$$f(x,y) = x^2$$

$$\vec{r}(t) = \langle \cos t, \sin t \rangle$$

$$\vec{r}'(t) = \langle -\sin t, \cos t \rangle$$

$$\|\vec{r}'(t)\| = \sqrt{\sin^2 t + \cos^2 t} = 1.$$

So area of wall

$$= \int_0^{\pi} (\cos t)^2 \cdot 1 \, dt$$

[Trig Identity?

$$\cos(2t) = \cos^2 t - \sin^2 t$$

$$\cos(2t) = \cos^2 t - (1 - \cos^2 t)$$

$$\cos(2t) = 2\cos^2 t - 1$$

$$2\cos^2 t = \cos(2t) + 1$$

$$\cos^2 t = \frac{1}{2}(1 + \cos(2t)).$$]

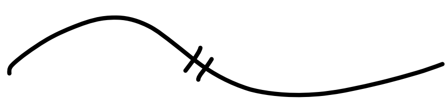
$$= \int_0^{\pi} \cos^2 t \, dt$$

$$= \int_0^{\pi} \frac{1}{2}\left(1 + \cos(2t)\right) dt$$

$$= \frac{1}{2}\left[t + \frac{1}{2}\sin(2t) \right]_0^{\pi}$$

$$= \frac{1}{2}\left[\pi + \frac{1}{2}\overset{0}{\cancel{\sin(2\pi)}} - 0 - \frac{1}{2}\underset{0}{\cancel{\sin(0)}} \right]$$

$$= \pi/2$$



Integrate scalar field over a surface in $\mathbb{R}^3$

$$\iint_{\text{surface}} \overbrace{f \, \underbrace{dA}_{\text{tiny piece of area in the surface}}}^{\text{tiny piece of mass}}$$

Typical: Surface area

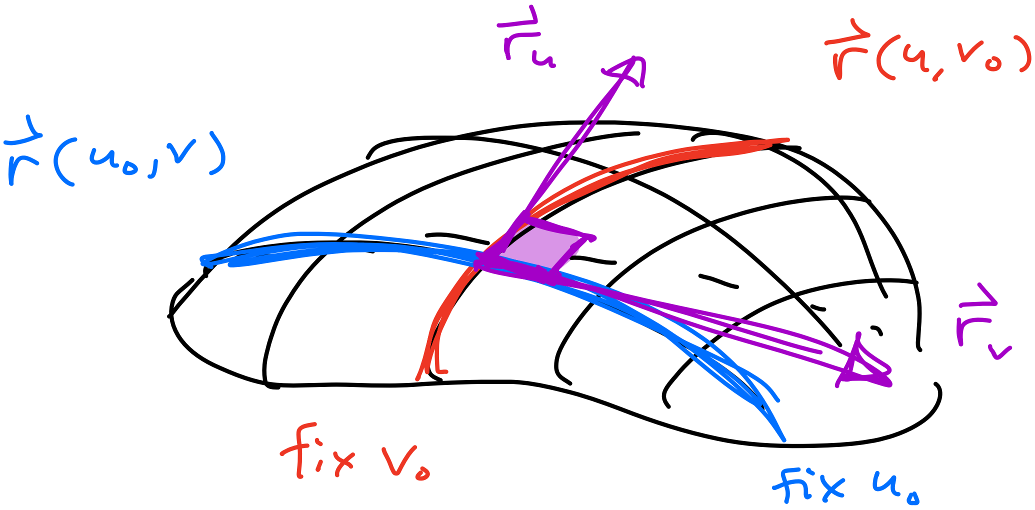
$$\iint_{\text{surface}} 1\, dA$$

add up all the tiny pieces of area.

How to compute?

Parametrize the surface:

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$



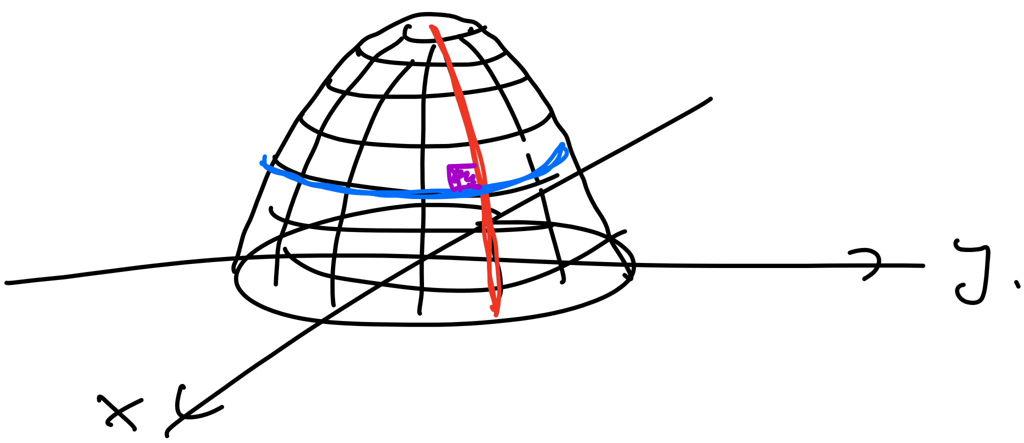
$$dA = \text{area of tiny parallelogram}$$

$$= \| \vec{r}_u \times \vec{r}_v \| \, du\, dv$$

So surface area is

$$\iint 1\, dA = \iint \|\vec{r}_u \times \vec{r}_v\|\, du\, dv.$$

Example: Surface area of parabolic dome $z = 1 - x^2 - y^2$, $x^2 + y^2 \leq 1$.



Parametrize the surface:

Use Polar in x,y-plane.

$$x = u\cos v \qquad 0 \leq u \leq 1$$

$$y = u\sin v \qquad 0 \leq v \leq 2\pi$$

$$[x^2 + y^2 = u^2]$$

$$z = 1 - x^2 - y^2$$
$$= 1 - u^2$$

$$\vec{r}(u,v) = \langle u\cos v,\ u\sin v,\ 1 - u^2 \rangle$$

$$\vec{r}_u = \langle \cos v, \sin v, -2u \rangle$$

$$\vec{r}_v = \langle -u\sin v, u\cos v, 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle 2u^2\cos v, 2u^2\sin v, u\cancel{\cos^2 v} + u\cancel{\sin^2 v} \rangle$$

$$= \langle 2u^2\cos v, 2u^2\sin v, u \rangle.$$

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{4u^4\cancel{\cos^2 v} + 4u^4\cancel{\sin^2 v} + u^2}$$

$$= \sqrt{4u^4 + u^2}$$

$$= \sqrt{u^2(4u^2+1)}$$

$$= u\sqrt{4u^2+1}$$

So surface area

$$= \iint \|\vec{r}_u \times \vec{r}_v\| \, du \, dv$$

$$= \iint u\sqrt{4u^2+1} \, du \, dv$$

$$= \int_0^{2\pi} dv \int_0^1 u\sqrt{4u^2+1}\, du$$

LUCKY !

$$= 2\pi \int_0^1 u\sqrt{4u^2+1}\, du$$

$w = 4u^2+1$

$dw = 8u\, du$

$u\, du = \frac{1}{8} dw$

$$= 2\pi \int_1^5 \frac{1}{8}\sqrt{w}\, dw$$

$$= \frac{\pi}{4}\left[\frac{2}{3} w^{3/2}\right]_1^5$$

$$= \frac{\pi}{6}\left[5^{3/2} - 1\right]$$

DONE.

General Pattern:

Parametrized k-dim "surface" living in n-dim space has form

$$\vec{r}(u_1,\dots,u_k) = \langle x_1(u_1,\dots,u_k), x_2(u_1,\dots,u_k), \dots, x_n(u_1,\dots,u_k)\rangle$$

Jacobian Matrix

$$J = \underbrace{\begin{pmatrix} (x_1)_{u_1} & \cdots & (x_1)_{u_k} \\ \vdots & & \vdots \\ (x_n)_{u_1} & \cdots & (x_n)_{u_k} \end{pmatrix}}_{k \text{ cols}} \left.\vphantom{\begin{pmatrix} a \\ b \\ c \end{pmatrix}}\right\} n \text{ rows}$$

To compute "k-volume" of this region:

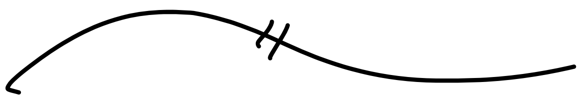
$$\int_{\text{region}} \underbrace{\sqrt{\det(J^T J)}}_{\substack{k\text{-volume} \\ \text{stretch factor}}} \, du_1 \, du_2 \cdots du_k$$

This is how the pattern continues:

$$\int_{\text{curve}} \|\vec{r}_u\| \, du$$

$$\iint_{\text{surface}} \|\vec{r}_u \times \vec{r}_v\| \, du \, dv$$

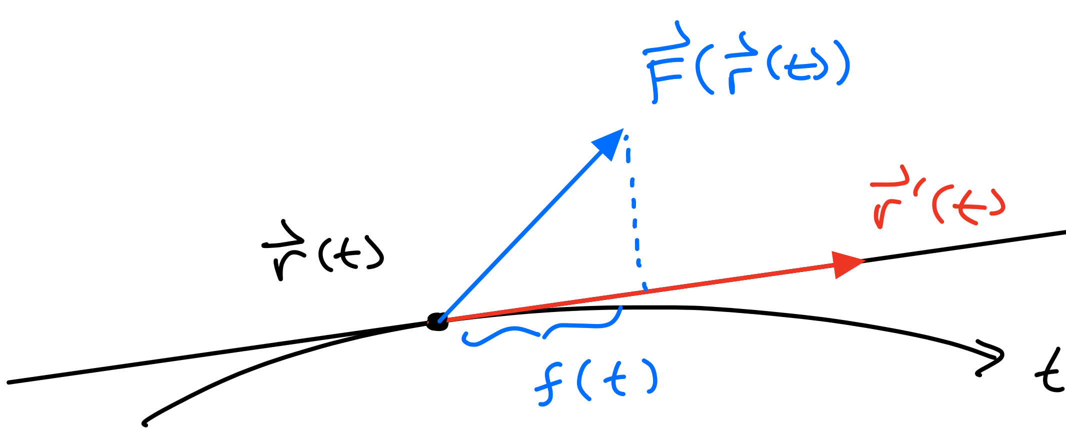
etc.



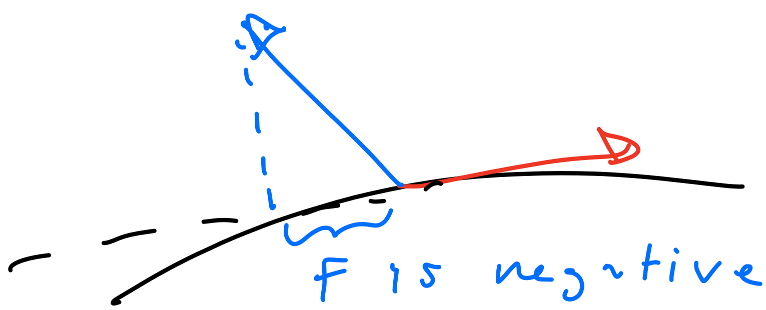
Next: Integrate <u>vector fields</u> over curves and surfaces.

WHY? Physics (Energy).

Think of particle moving in a force field (e.g. gravity).



Let $f(t)$ be the component of the force $\vec{F}(\vec{r}(t))$ in the direction of the velocity $\vec{r}'(t)$, so $f(t)$ is a scalar. It can be negative when force opposes motion:



From physics

$$\int_{\text{curve}} f\, ds$$

$$\int_{\text{curve}} f(t)\, \|\vec{r}'(t)\|\, dt$$

= amount of Kinetic energy added to particle by force field.

Could be negative. Friction always resists the motion, so $f(t) < 0$

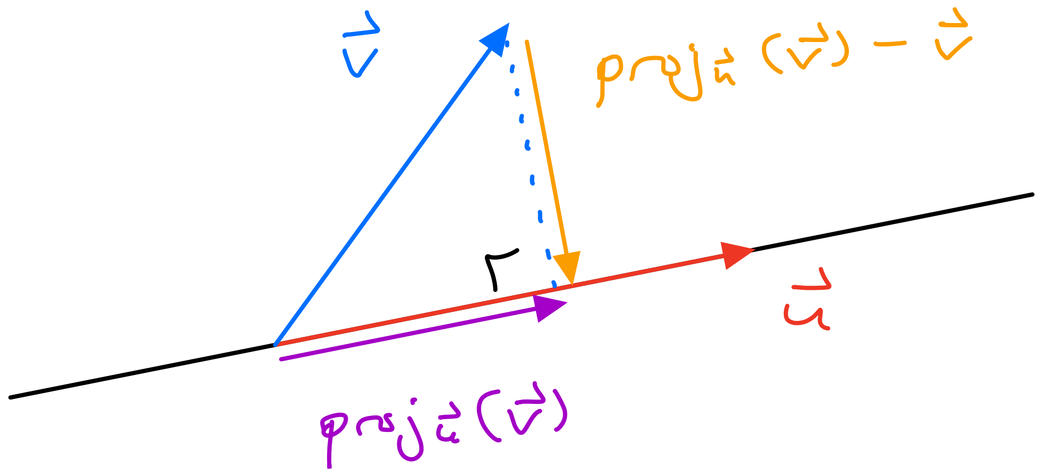
Change in kinetic energy due to friction $= \displaystyle\int \underbrace{f(t)\,\|\vec{r}\,'(t)\|}_{\text{always } < 0}\, dt < 0.$

Let's be precise. Consider a force field & parametrized curve:

$$\vec{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$\vec{r} : \mathbb{R} \longrightarrow \mathbb{R}^3$$

We need a formula for the component of $\vec{F}(\vec{r}(t))$ in the direction of $\vec{r}\,'(t)$.

Projection:



Formula? TWO FACTS:

(1) $\text{proj}_{\vec{u}}(\vec{v}) = \alpha \vec{u}$ for some scalar α.

(2) There is a right angle, i.e., the dot product of two vectors is zero. Which two vectors?

$$\vec{u} \cdot \left(\text{proj}_{\vec{u}}(\vec{v}) - \vec{v} \right) = 0.$$

Put (1) & (2) together:

$$\vec{u} \cdot (\alpha \vec{u} - \vec{v}) = 0$$

$$\alpha\, \vec{u} \cdot \vec{u} - \vec{u} \cdot \vec{v} = 0$$

$$\alpha = \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}$$

Conclusion:

$$\boxed{\operatorname{proj}_{\vec{u}}(\vec{v}) = \underbrace{\frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}}_{\text{scalar}} \underbrace{\vec{u}}_{\text{vector}}}$$

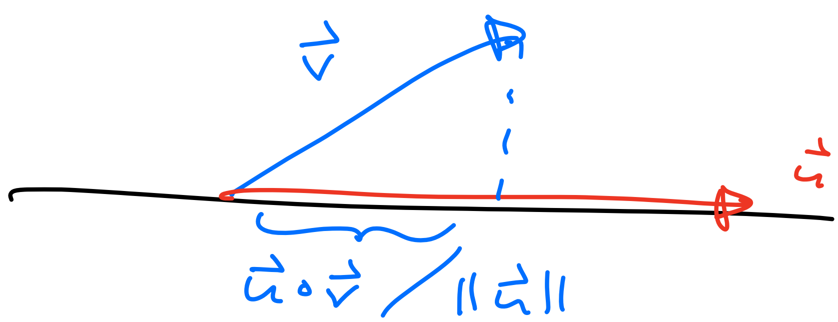
The length of the projection?

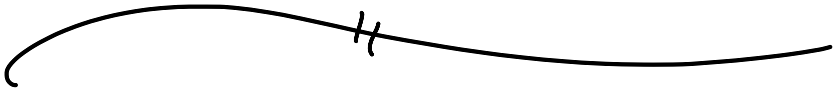
$$\|\operatorname{proj}_{\vec{u}}(\vec{v})\| = \left|\frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\right| \|\vec{u}\|$$

$$= \frac{|\vec{u} \cdot \vec{v}|}{\|\vec{u}\|^2} \|\vec{u}\|$$

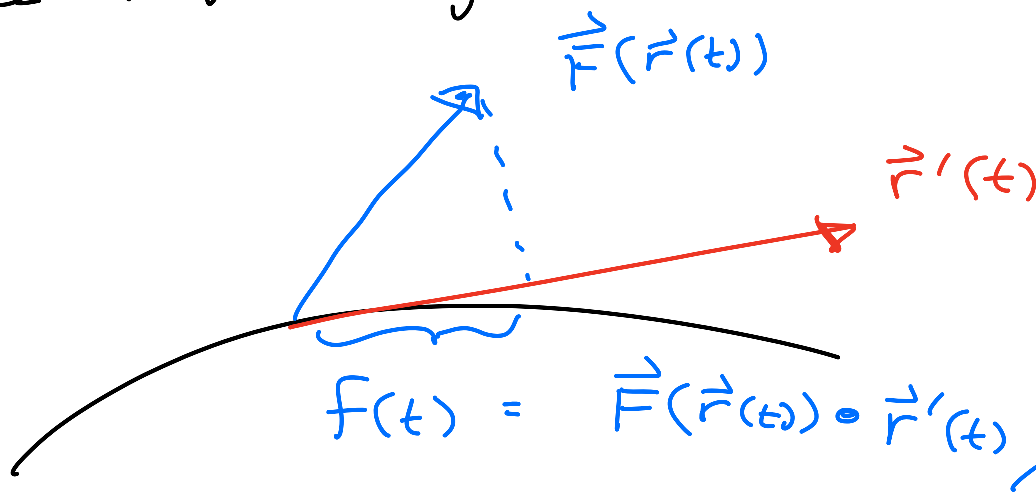
$$= |\vec{u} \cdot \vec{v}| \,/\, \|\vec{u}\|$$

If we want to allow negatives:





Force & Velocity :



$$f(t) = \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \Big/ \|\vec{r}\,'(t)\|$$

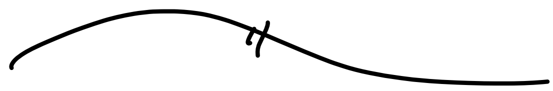
Finally : Work done by a changing force on a moving particle

$$= \int_{\text{curve}} f(t)\, \|\vec{r}\,'(t)\|\, dt$$

$$= \int_{\text{curve}} \frac{\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)}{\cancel{\|\vec{r}\,'(t)\|}} \cdot \cancel{\|\vec{r}\,'(t)\|}\, dt$$

$$= \int \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int (\text{force}) \cdot (\text{velocity})\, dt$$



Most Interesting Example:

Gravity.

More generally, say a force field

$$\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$$

is "conservative" if it is the gradient of some scalar field:

$$\vec{F} = \nabla f$$

$$\left(\vec{F} = -\nabla f \text{ in physics} \right).$$

We will see that conservative force fields have many special properties.

One property is the fundamental

Theorem:

If $\vec{F} = \nabla f$ then

$$\int_a^b \vec{F}(\vec{r}(t)) \circ \vec{r}\,'(t)\, dt$$

$$= f(\vec{r}(b)) - f(\vec{r}(a))$$

Proof: "Completely easy"

Chain rule:

$$[f(\vec{r}(t))]' = \nabla f(\vec{r}(t)) \bullet \vec{r}\,'(t).$$

Use Fundamental Theorem from Calc I & II:

dot product.

$$\int_a^b \vec{F}(\vec{r}(t)) \bullet \vec{r}\,'(t)\, dt$$

Calc I

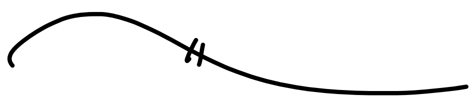
$$= \int_a^b [f(\vec{r}(t))]'\, dt$$

$$= f(\vec{r}(b)) - f(\vec{r}(a))$$

[Recall: For any $g(t)$ we have

$$\int_a^b g'(t)\,dt = g(b) - g(a).$$

Here we take $g(t) = f(\vec{r}(t))$.]



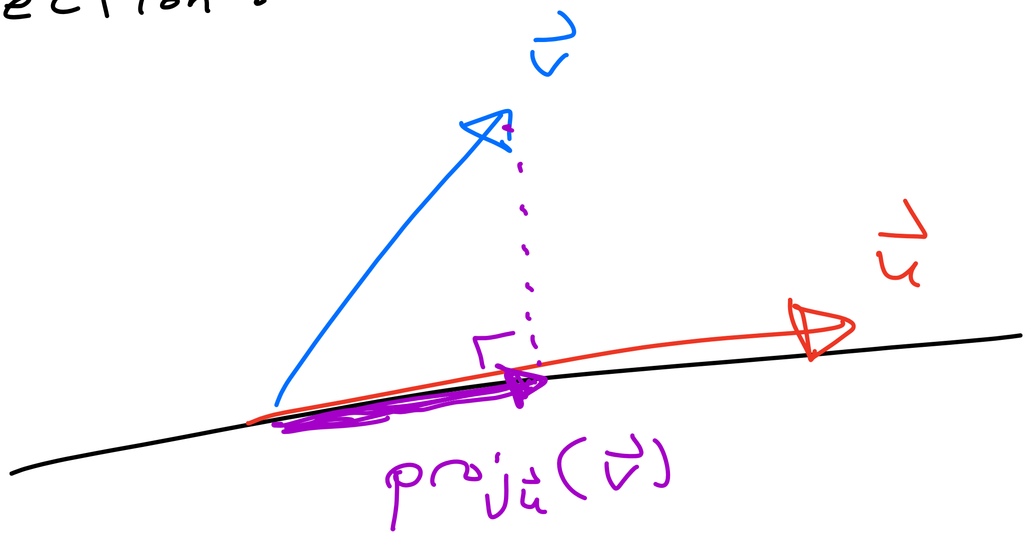
Interpretation:

Kinetic Energy + Potential Energy = Constant.

HW 5 is posted; due on <u>Tues</u>.

Quiz 5 next Wed.

Projection:



Formula:

$$\text{proj}_{\vec{u}}(\vec{v}) = \underbrace{\frac{\vec{u}\circ\vec{v}}{\vec{u}\circ\vec{u}}}_{\text{scalar}} \underbrace{\vec{u}}_{\text{vector}}$$

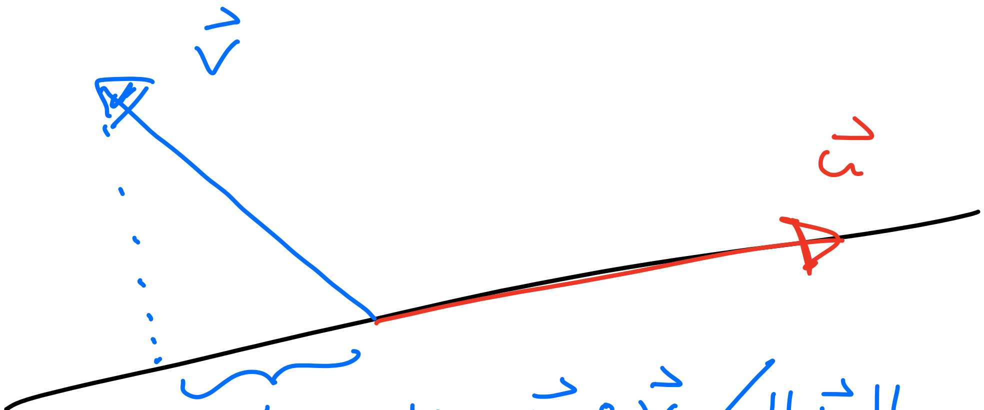
$$= \frac{\vec{u}\circ\vec{v}}{\|\vec{u}\|^2}\cdot\vec{u}$$

$$= \frac{\vec{u}\circ\vec{v}}{\|\vec{u}\|}\cdot\frac{\vec{u}}{\|\vec{u}\|}$$

a unit vector in the direction of $\vec{u}$.

The magnitude (positive or negative) of the projection is $\vec{u} \cdot \vec{v} / \|\vec{u}\|$.

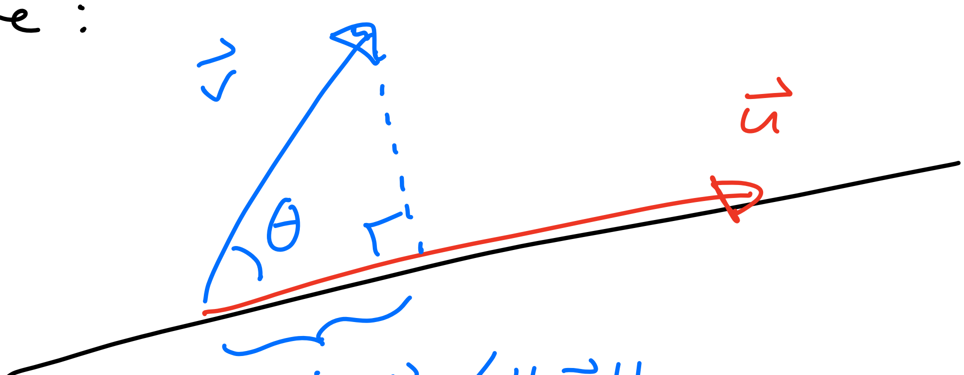
[Special Case $\|\vec{u}\|$ is nicest.]



Length $\vec{u} \cdot \vec{v} / \|\vec{u}\|$
(could be negative)

Call $\vec{u} \cdot \vec{v} / \|\vec{u}\|$ the "component of $\vec{v}$ in the direction of $\vec{u}$"

Picture:



$\vec{u} \cdot \vec{v} / \|\vec{u}\|$

$\|\vec{u}\| \cdot \|\vec{v}\| \cos\theta / \|\vec{u}\|$

$\|\vec{v}\| \cos\theta$

Example: Project $\vec{v} = \langle 1, 0, 0 \rangle$ onto $\vec{u} = \langle 1, 1, 1 \rangle$

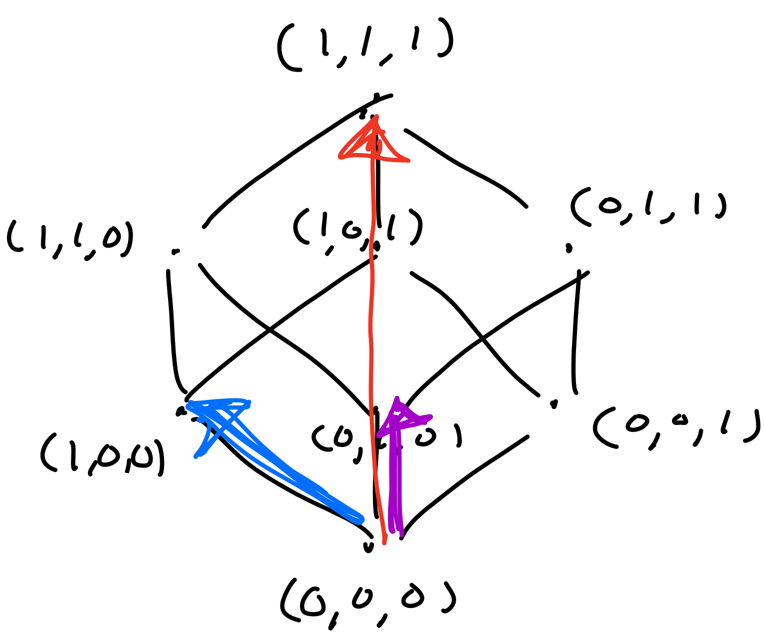
$$\mathrm{proj}_{\vec{u}}(\vec{v}) = \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \vec{u}$$

$$= \frac{1+0+0}{1+1+1} \langle 1, 1, 1 \rangle$$

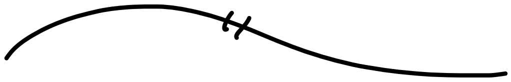
$$= \frac{1}{3} \langle 1, 1, 1 \rangle$$

$$= \left\langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right\rangle$$

Picture: A Cube sitting on corner.

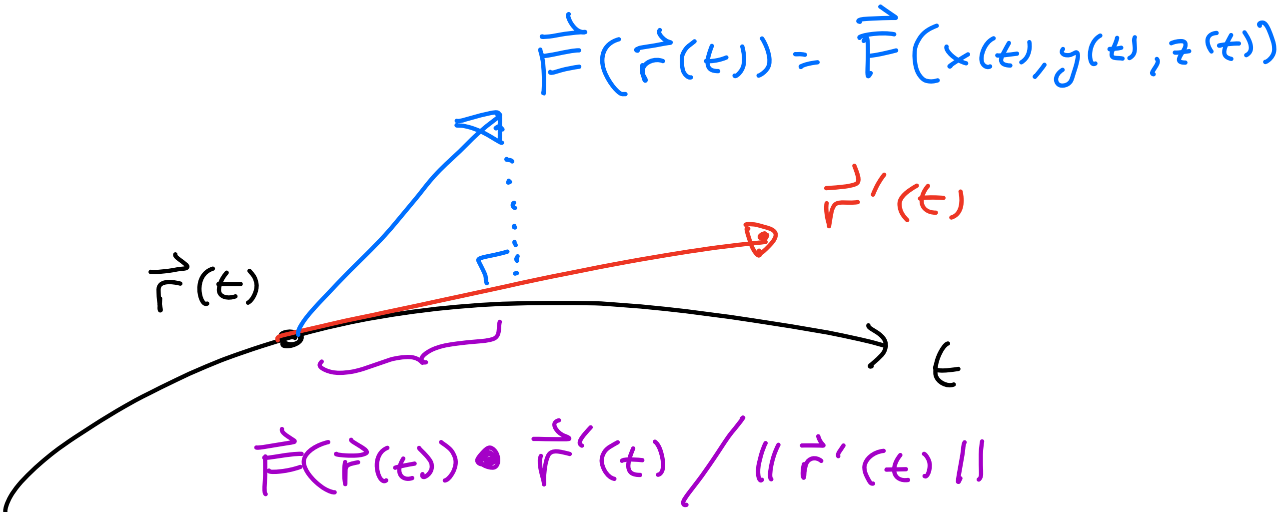


projection of (1,0,0) on (1,1,1) is $\frac{1}{3}$ of the way up the cube.



We use projection to define the integral of a vector field along a parametrized curve.

Consider vector field $\vec{F}(x,y,z)$ and curve $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$.



$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \,/\, \|\vec{r}\,'(t)\|$

component of vector field $\vec{F}$ in direction of the curve $\vec{r}$.

Define integral of $\vec{F}$ along $\vec{r}$ as the integral of this component:

$$\int_{\text{curve}} \underbrace{\frac{\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}}_{\text{scalar}} \underbrace{\|\vec{r}\,'(t)\|\, dt}_{ds}$$

If we don't want to mention the parametrization, we can write

$$\int_C \vec{F} \cdot \underbrace{\vec{T}}_{\substack{\text{unit vector} \\ \text{tangent to curve.}}} \, ds$$

After parametrizing we get

$$\vec{T} = \underbrace{\vec{r}\,'(t) \,/\, \|\vec{r}\,'(t)\|}_{\text{unit vector tangent to curve.}}$$

$$ds = \underbrace{\|\vec{r}\,'(t)\|\, dt}_{\text{tiny piece of arc length}}$$

That's a lot of Jargon!

MEANING:

$$\int_C \vec{F} \cdot \vec{T} \, ds = \text{“how much on average does } \vec{F} \text{ point along the curve”?}$$

Physics:

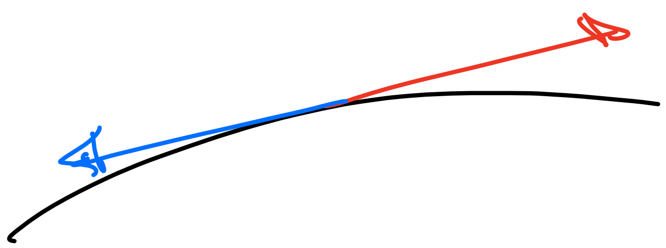
$$\int \vec{F}(\vec{r}(t)) \circ \vec{r}\,'(t)\, dt$$

= "how much work done on particle $\vec{r}(t)$ by force field $\vec{F}$"?

= "kinetic energy added to the particle by force field."

e.g. If $\vec{F}$ is friction then we always have $\vec{F}(\vec{r}(t)) \circ \vec{r}\,'(t) < 0$

[force opposes the motion, i.e., is in the opposite direction from your velocity:



]

In this case

$$\int \underbrace{\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)}_{\text{always} < 0} \, dt < 0$$

This force <u>decreases</u> your KE.

Example: Gravity near surface of the Earth.

Pick coordinates so

z-axis points "up"

$z = 0$ is ground level.

Particle of mass m.

Launch the particle directly up with speed v.

$$\vec{r}(0) = \langle 0, 0, 0 \rangle$$

$$\vec{r}\,'(0) = \langle 0, 0, v \rangle.$$

$$\vec{r}\,''(t) = \langle 0, 0, -32 \text{ ft/sec}^2 \rangle.$$

Integrate:

$$\vec{r}\,'(t) = \langle \underbrace{\cancel{c_1}}_{0}, \underbrace{\cancel{c_2}}_{0}, -32t + \underbrace{\cancel{c_3}}_{v} \rangle$$

$$\vec{r}\,'(t) = \langle 0, 0, -32t + v \rangle$$

$$\vec{r}(t) = \langle \underbrace{\cancel{c_4}}_{0}, \underbrace{\cancel{c_5}}_{0}, -16t^2 + vt + \underbrace{\cancel{c_6}}_{0} \rangle$$

$$\vec{r}(t) = \langle 0, 0, -16t^2 + vt \rangle.$$

The force satisfies Newton's 2nd:

$$\vec{F}(t) = m\,\vec{r}\,''(t).$$

$$= m \langle 0, 0, -32 \rangle$$

$$= \underbrace{\langle 0, 0, -32m \rangle}_{\text{constant vector.}}$$

KEY Property of Gravity:

It has an anti-derivative, meaning if $\vec{F}(x,y,z)$ is the gravitational force, then <u>we can find</u> a scalar field $f(x,y,z)$

such that

$$\vec{F}(x,y,z) = \nabla f(x,y,z).$$

[Jargon: Vector field $\vec{F}$ with an anti-deriv $\vec{F} = \nabla f$ is called a "conservative vector field".]

For us:

$$\vec{F}(x,y,z) = \langle 0, 0, -32m \rangle$$

at any point (x,y,z). Look for an anti-derivative $f(x,y,z)$.

$$\vec{F} = \nabla f$$

$$\langle 0, 0, -32m \rangle = \langle f_x, f_y, f_z \rangle.$$

$f_z = -32m \rightsquigarrow f = -32mz$ + something that does not involve z.

$$f(x,y,z) = -32mz + g(x,y)$$

for some function $g(x,y)$.

NEXT: $f_x = 0$.

$$\frac{d}{dx}\left(-32mz + g(x,y)\right) = 0$$

$$0 + g_x = 0$$

$$g_x = 0$$

$$g(x,y) = h(y).$$

for some function $h(y)$ of y.

Currently: $f(x,y,z) = -32mz + h(y)$.

FINALLY: $f_y = 0$.

$$\frac{d}{dy}\left(-32mz + h(y)\right) = 0$$

$$0 + h'(y) = 0$$

$$h(y) = c$$

for some constant c.

Conclusion: $f(x,y,z) = -32mz + c$.

For physical reasons, want

$$\vec{F} = -\nabla f$$

so take $f(x,y,z) = +32mz + c$.

If $\vec{F}$ is a force field

& $\vec{F} = -\nabla f$ then

f is called "potential energy".

In our case:

$\vec{F}(x,y,z) = \langle 0, 0, -32m \rangle$

= force of gravity acting on a particle of mass m at point (x,y,z).

$f(x,y,z) = +32mz + c$

= gravitational potential of a particle of mass m at point (x,y,z).

Fundamental Theorem of "Line Integrals" (i.e. integrals of vector fields along curves).

$$\int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = f(\vec{r}(b)) - f(\vec{r}(a))$$

$$\int_{\text{along curve}} \nabla f = f(\text{endpoint}) - f(\text{start point})$$

e.g. $f(x,y,z) = xyz$

$$\nabla f(x,y,z) = \langle yz, xz, xy \rangle$$

Integrate along some curve:

$$\vec{r}(t) = \langle t, t^2, t^3 \rangle$$

$t = 1$ to $t = 2$.

Prediction: $$\int_{\text{curve}} \nabla f = f(\vec{r}(2)) - f(\vec{r}(1)).$$

$$= f(2,4,8) - f(1,1,1)$$

$$= 2\cdot 4\cdot 8 - 1\cdot 1\cdot 1 = 63.$$

Check:

$$\vec{F} = \nabla f = \langle yz, xz, xy \rangle.$$

$$\int_{\text{curve}} \vec{F} = \int \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int \langle t^2\cdot t^3, t\cdot t^3, t\cdot t^2 \rangle \cdot \langle 1, 2t, 3t^2 \rangle\, dt$$

$$= \int_1^2 \left(t^5 + 2t^5 + 3t^5 \right) dt$$

$$= \int_1^2 6t^5\, dt$$

$$= 6 \cdot \frac{1}{6} t^6 \Big|_1^2$$

$$= 2^6 - 2^1 = 63 \checkmark$$

Proof of F.T.L.I.

Chain Rule:

$$\frac{d}{dt}\left(f(\vec{r}(t)) \right)$$

$$= \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

Integrate both sides with resp. to t.

Let $g(t) = f(\vec{r}(t))$.

$$\int \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt.$$

$$= \int_a^b \frac{d}{dt} g(t)\, dt$$

Calc I

$$= g(b) - g(a) \checkmark$$

Physics: Let $\vec{F}$ be force field. Suppose $\vec{F} = -\nabla f$ for some scalar field f (called the "potential energy"). Then

$$\underbrace{\underbrace{\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt}_{\text{increase in KE}} = \underbrace{-f(\vec{r}(b)) + f(\vec{r}(a))}_{\text{decrease in PE}}}_{\text{conservation of mechanical energy}}.$$

Back to our example:

$$\vec{F}(x,y,z) = \langle 0, 0, -32m \rangle$$

$$f(x,y,z) = +32mz + c$$

Let's choose c so potential

energy is zero on the ground.

$\rightsquigarrow$ $c = 0$.

$f(x, y, 0) = 0$.

$$PE(t) = f(\vec{r}(t))$$

$$= f(0, 0, -16t^2 + vt)$$

$$= +32m(-16t^2 + vt)$$

$$= -512mt^2 + 32mvt$$

Define the kinetic energy at time t:

$$KE(t) = \frac{1}{2} m \|\text{velocity}\|^2$$

$$= \frac{1}{2} m \|\vec{r}\,'(t)\|^2$$

$$= \frac{1}{2} m \|\langle 0, 0, -32t + v \rangle\|^2$$

$$= \frac{1}{2} m (-32t + v)^2$$

$$= \frac{1}{2} m (1024t^2 - 64vt + v^2)$$

$$= 512mt^2 - 32mvt + \frac{1}{2}mv^2$$

Conclusion:

$$KE(t) + PE(t) = \frac{1}{2}mv^2$$

constant, i.e., independent of t.

At time $t = 0$ we have

$$KE(0) = \frac{1}{2}mv^2$$

$$PE(0) = 0$$

When the particle reaches the top, it has no velocity, so $KE(\text{top}) = 0$.

Hence

$$KE(\text{top}) = 0$$

$$PE(\text{top}) = \frac{1}{2}mv^2.$$

$$+32mz = \frac{1}{2}mv^2$$

$$z = \frac{1}{64}v^2$$

This is how high the particle will go. We could have solved this by maximizing the z coord:

$$z(t) = -16t^2 + vt.$$

But I wanted to illustrate the concept of potential energy, which applies in much more general situations.

HW 5 due Tues

Quiz 5 on Wed

Final Project due next Fri June 24.

Now: Chapter 6 (Vector Calculus)

Recall: Given vector field

$$\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$$

and a curve $\vec{r} : \mathbb{R} \to \mathbb{R}^n$, we define the "line integral"

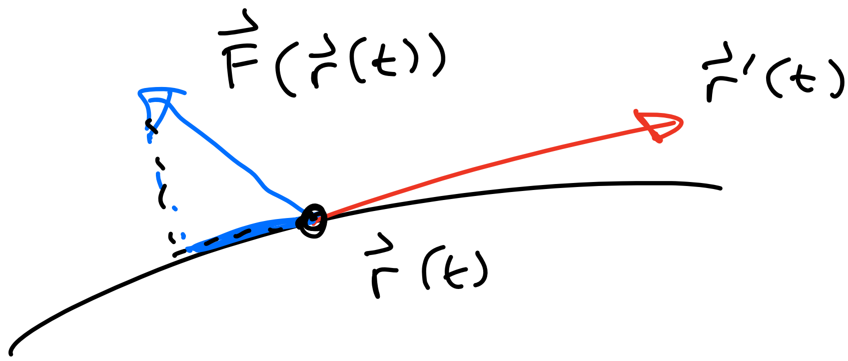
$$\int_{\text{curve}} \vec{F} = \int \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

= sum the component of $\vec{F}$ along the curve

= "on average, how much does $\vec{F}$ point in the direction of the curve?"

$= 0$ if $\vec{F} \perp$ curve at every point

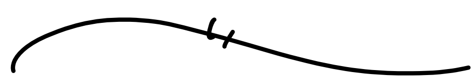
< 0 if $\vec{F}$ points against the curve.



here $\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) < 0$

Physics: $\vec{F}$ force field.

$\int_{\text{curve}} \vec{F}$ = amount of KE added to particle by the field.
("speed")



Fund Thm Line Integrals:

If $\vec{F} = \nabla f$ then

$$\int_{\text{curve}} \vec{F} = f(\text{end point}) - f(\text{start point})$$

Proof :

$$\int_{\text{curve}} \vec{F} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

CHAIN RULE

$$= \int_a^b \frac{d}{dt}\left[f(\vec{r}(t)) \right] dt$$

Calc I

$$= f(\vec{r}(b)) - f(\vec{r}(a)) \quad \checkmark$$

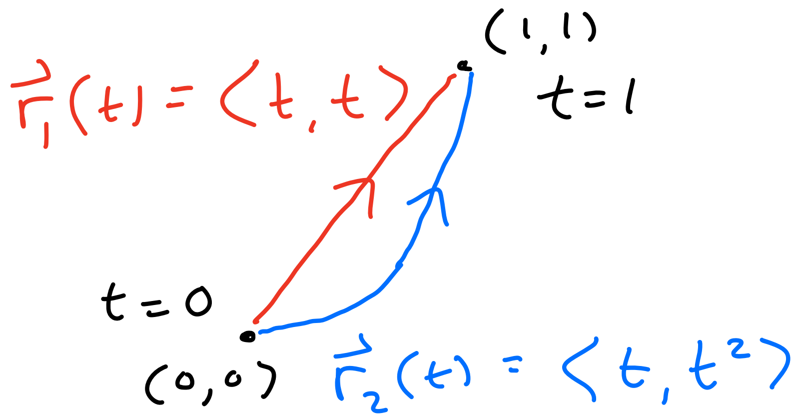
Consequence : If $\vec{F} = \nabla f$ then

$\int_{\text{curve}} \vec{F}$ only depends on the endpoints, <u>not</u> on the shape of the curve.

Example:

$$\vec{F} = \nabla(xy + y)$$

$$= \langle y, x+1 \rangle$$



$$\int_0^1 \vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1(t)\, dt$$

$$= \int \langle t, t+1 \rangle \cdot \langle 1, 1 \rangle\, dt$$

$$= \int (t + (t+1))\, dt$$

$$= \int (2t+1)\, dt$$

$$= \left[2 \cdot \frac{t^2}{2} + t \right]_0^1$$

$$= 1 + 1 = 2.$$

$$\int_0^1 \vec{F}(\vec{r}_2(t)) \cdot \vec{r}_2\,'(t)\, dt$$

$$= \int \langle t^2, t+1 \rangle \cdot \langle 1, 2t \rangle \, dt$$

$$= \int \left[t^2 + (t+1)(2t) \right] dt$$

$$= \int \left(t^2 + 2t^2 + 2t \right) dt$$

$$= \int \left(3t^2 + 2t \right) dt$$

$$= \left[3 \cdot \frac{t^3}{3} + 2 \cdot \frac{t^2}{2} \right]_0^1$$

$$= 1 + 1 = 2.$$ SAME ✓

In fact:

$$\int_{\text{curve}} \vec{F} = f(\text{end point}) - f(\text{start})$$

$$= f(1,1) - f(0,0)$$

$$= (1\cdot 1 + 1) - (0\cdot 0 + 0)$$

$$= 2.$$

That's why the two paths give the same answer.

Now let's change $\vec{F}$ a little bit

$$\vec{F}(x,y) = \langle y, x+1 \rangle$$

$$\vec{G}(x,y) = \langle y, 2x+1 \rangle$$

Integrate $\vec{G}$ along the two paths.

$$\int_0^1 \vec{G}\big(\underset{t,\,t}{\cancel{\vec{r}_1(t)}}\big) \cdot \underset{1,\,1}{\cancel{\vec{r}_1{}'(t)}}\, dt$$

$$= \int \langle t, 2t+1 \rangle \cdot \langle 1, 1 \rangle \, dt$$

$$= \int \big(t + (2t+1)\big)\, dt$$

$$= \int (3t + 1)\, dt$$

$$= \left(3 \cdot \frac{t^2}{2} + t\right)\Big|_0^1$$

$$= \frac{3}{2} + 1 = \frac{5}{2}.$$

$$\int_0^1 \vec{G}(\cancel{\vec{r}_2(t)}) \circ \vec{r}_2{}'(t)\, dt$$

$$\langle t, t^2 \rangle \qquad \langle 1, 2t \rangle$$

$$= \int \langle t^2, 2t+1 \rangle \bullet \langle 1, 2t \rangle \, dt$$

$$= \int \left(t^2 + (2t+1)(2t)\right) dt$$

$$= \int \left(t^2 + 4t^2 + 2t\right) dt$$

$$= \int (5t^2 + 2t)\, dt$$

$$= \left[5 \cdot \frac{t^3}{3} + 2 \cdot \frac{t^2}{2}\right]_0^1$$

$$= \frac{5}{3} + 1 = \frac{8}{3} \neq \frac{5}{2}$$

NOT THE SAME !

Today we'll discuss what went wrong.

But first, Kinetic Energy.

Consider a moving particle $\vec{r}(t)$ with mass m. Define

$$KE(t) = \underbrace{\frac{1}{2} m \|\vec{r}\,'(t)\|^2}_{\text{WHY ?}}$$

Suppose force field $\vec{F}$ acts on the particle, so $\vec{F}(\vec{r}(t)) = m\vec{r}\,''(t)$.

Compute $KE'(t)$.

$$KE(t) = \frac{1}{2} m \|\vec{r}\,'(t)\|^2$$

$$= \frac{1}{2} m \underbrace{\vec{r}\,'(t) \cdot \vec{r}\,'(t)}$$

Product Rule

$$KE'(t) = \frac{1}{2}m\left[\vec{r}''(t)\cdot\vec{r}'(t) + \vec{r}'(t)\cdot\vec{r}''(t)\right]$$

$$= \frac{1}{2}m\left[2\vec{r}''(t)\cdot\vec{r}'(t)\right]$$

$$= \underbrace{m\,\vec{r}''}(t)\cdot\vec{r}'(t)$$

$$= \underbrace{\vec{F}(\vec{r}(t))\cdot\vec{r}'(t)}_{\text{what do we see?}}.$$

KE'(t) looks familiar!

$$KE(t) = \int \vec{F}(\vec{r}(t))\cdot\vec{r}'(t)\,dt$$

$$\int_{\text{curve}} \vec{F}\text{orce} = KE(\text{end}) - KE(\text{start})$$

$$= \text{increase in } KE$$

Applies for ANY force field.

Now, assume $\vec{F}$ is conservative:

$$\vec{F} = -\nabla f \text{ for some } f.$$

Then we also have

$$\int_{\text{curve}} \vec{F} = \int -\nabla f$$

$$= -\int \nabla f$$

$$= -\left[f(\text{end}) - f(\text{start}) \right]$$

$$= f(\text{start}) - f(\text{end})$$

↑

Fund Thm Line Integrals

So let's define the potential energy

$$PE(t) = f(\vec{r}(t)).$$

Then combining the above equations:

$$KE(\text{end}) - KE(\text{start})$$
$$= PE(\text{start}) - PE(\text{end}).$$

$$KE(\text{start}) + PE(\text{start})$$
$$= KE(\text{end}) + PE(\text{end}).$$

"Conservation of Mechanical Energy"

Energy is converted between KE & PE but never destroyed.

This is <u>why</u> gradient vector fields are called "conservative".

Example: Gravity near planet.

$$\vec{F}(x,y,z) = \langle 0, 0, -mg \rangle$$
$$\vec{r}(0) = \langle 0,0,0 \rangle$$
$$\vec{r}\,'(0) = \langle 0,0,v \rangle \quad \text{up. } (v>0)$$
$$m\vec{r}\,''(t) = \vec{F}(\vec{r}(t))$$
$$m\vec{r}\,''(t) = \langle 0,0,-mg \rangle$$
$$\vec{r}\,''(t) = \langle 0,0,-g \rangle \quad \text{constant.}$$
$$\vec{r}\,'(t) = \langle 0,0,-gt+v \rangle$$
$$\vec{r}(t) = \langle 0,0,-\tfrac{1}{2}gt^2+vt \rangle$$

$$KE(t) = \frac{1}{2}m\|\vec{r}\,'(t)\|^2$$

$$= \frac{1}{2} m \left[0^2 + 0^2 + (-gt + v)^2 \right]$$

$$= \frac{1}{2} m \left[g^2 t^2 - 2gvt + v^2 \right]$$

$$= \frac{1}{2} m g^2 t^2 - mgvt + \frac{1}{2} m v^2 .$$

Next: Observe that $\vec{F}$ is conservative.

$$f(x,y,z) = mgz$$

$$-\nabla f = \langle 0, 0, -mg \rangle = \vec{F} .$$

Define

$$PE(t) = f(\vec{r}(t)) .$$

$$= f(0, 0, -\tfrac{1}{2} g t^2 + vt)$$

$$= mg\left(-\tfrac{1}{2} g t^2 + vt\right)$$

$$= -\frac{1}{2} m g^2 t^2 + mgvt$$

Finally we have

$$KE(t) + PE(t) = \underbrace{\frac{1}{2}mv^2}_{\text{independent of } t.}$$

$$PE(\text{start}) = f(0,0,0) = 0$$

$$KE(\text{start}) = \frac{1}{2}m\|\vec{r}\,'(0)\|^2 = \frac{1}{2}mv^2$$

When the projectile reaches max height we get $\|\vec{r}\,'(t)\| = 0$, so $KE(\text{top}) = 0$.

$$PE(\text{top}) = \frac{1}{2}mv^2 - KE(\text{top})$$

$$PE(\text{top}) = \frac{1}{2}mv^2$$

$$\cancel{m}g\,z(\text{top}) = \frac{1}{2}\cancel{m}v^2$$

$$z(\text{top}) = \frac{1}{2g}v^2$$

This is the max height of the particle. Note: It is independent of mass!

UNITS :

$g \sim$ accel $\sim m/s^2$

$v \sim$ velocity $\sim m/s$

$$\frac{1}{2g} \cdot v^2 \sim \frac{1}{m/s^2} \cdot \left(\frac{m}{s}\right)^2 \sim m$$

So $\frac{1}{2g} v^2 \sim$ Length ✓

Back to Math.

Since $\vec{G} = \langle y, 2x+1 \rangle$ does not satisfy "independence of path", it cannot be a gradient vector field.

Is there an easier way to see this?

Theorem (Conservative Vector Fields).

Given vector field in $\mathbb{R}^2$:

$$\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$$

The following statements are equivalent.

- $\vec{F} = \nabla f$ for some $f(x,y)$
- $\int_{\text{Loop}} \vec{F} = 0$ for any Loop
- "Cross-Partial Property"

$$P_y = Q_x$$

Check: $\vec{F}(x,y) = \langle y, x+1 \rangle$

$$P(x,y) = y$$

$$Q(x,y) = x+1$$

$$P_y = 1$$

$$Q_x = 1$$

SAME

So $\vec{F}$ is conservative.

But $\vec{G}(x,y) = \langle y, 2x+1 \rangle$

$$P_y = 1$$

$$Q_x = 2$$

NOT SAME

So $\vec{G}$ is <u>not</u> conservative.

3D Version : Given

$$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$$

The following are equivalent:

- $\vec{F} = \nabla f$ for some $f(x,y,z)$

- $\int_{Loop} \vec{F} = 0$ for any Loop.

- $\begin{cases} P_y = Q_x \\ P_z = R_x \\ Q_z = R_y \end{cases}$ "cross-partial property"

[In Higher Dimensions:

$$\vec{F}(x_1, \dots, x_n) = \langle F_1(x_1, \dots, x_n), \dots, F_n(x_1, \dots, x_n) \rangle$$

Cross-Partial property says

$$\frac{dF_i}{dx_j} = \frac{dF_j}{dx_i} \text{ for all } i \neq j.$$

EASY TO CHECK :)]

Example: $\quad P \quad Q \quad R$

$$\vec{F}(x,y,z) = \langle 3x^2 z,\ z^2,\ x^3 + 2yz \rangle$$

Check cross partials:

$P_y = 0$ & $Q_x = 0$ ✓

$P_z = 3x^2$ & $R_x = 3x^2$ ✓

$Q_z = 2z$ & $R_y = 2z$ ✓

This guarantees that $\vec{F}$ has an antiderivative scalar field.

How can we find it?

TWO METHODS:

(1) Try really hard.

Looking for $f(x,y,z)$ such that

$$f_x(x,y,z) = 3x^2 z$$
$$f_y(x,y,z) = z^2$$
$$f_z(x,y,z) = x^3 + 2yz$$

START:

$f_y = z^2$

$f = z^2 y + g(x,z)$

$f_x = 3x^2 z$

$f_x = 0 + g_x$

$g_x = 3x^2 z$

$g = x^3 z + h(y,z)$

Seems like we're going around in circles!

(2) Use the Fund Thm:

If $\vec{F} = \nabla f$ then

$$\int_{curve} \vec{F} = f(\text{end}) - f(\text{start}).$$

(Independent of the shape of curve.).

TRICK: Fix some start point

start $= (0, 0, 0)$

Consider <u>any</u> path from $(0,0,0)$ to some point (a,b,c).

Say $\vec{r}(t) = (at, bt, ct)$

$t = 0$ to 1.

Then

$$\int_{\text{curve}} \vec{F} = \underbrace{f(a,b,c)}_{\text{this is what we want to know.}} - \underbrace{f(0,0,0)}_{\text{const.}}.$$

So let's compute:

$$\int_0^1 \vec{F}(at, bt, ct) \cdot \langle a, b, c\rangle \, dt$$

$$= \int_0^1 \left\langle 3(at)^2(ct), (ct)^2, (at)^3 + 2(bt)(ct) \right\rangle \cdot \langle a, b, c \rangle \, dt.$$

$$= \int 3a^3ct^3 + bc^2t^2 + ca^3t^3 + 2bc^2t^2 \, dt$$

$$= 3a^3c\frac{t^4}{4} + bc^2\frac{t^3}{3} + ca^3\frac{t^4}{4} + 2bc^2\frac{t^3}{3} \Big|_0^1$$

$$= \underbrace{\frac{3}{4}a^3c + \frac{bc^2}{3} + \frac{ca^3}{4} + \frac{2bc^2}{3}}$$

This is our desired $f(a,b,c)$.

In other words:

$$f(x,y,z) = \frac{3}{4}x^3z + \frac{1}{3}yz^2 + \frac{1}{4}x^3z + \frac{2}{3}yz^2.$$

$$= x^3z + yz^2$$

CHECK:

$$f(x,y,z) = x^3z + yz^2$$

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

$$= \langle 3x^2z,\ z^2,\ x^3 + 2yz \rangle$$

$$= \vec{F} \quad \checkmark$$

It worked.