Math 310 — Fall 2023
Homework 6 — Drew Armstrong

1. Various Kinds of First and Second Derivatives in $\mathbb{R}^3$. For any scalar field $f(x,y,z)$ we define a vector field grad(f) and a scalar field laplacian(f) by

$$\begin{aligned}\text{grad}(f) &= \text{“}\nabla f\text{”} = \langle f_x, f_y, f_z\rangle,\\ \text{laplacian}(f) &= \text{“}\nabla^2 f\text{”} = f_{xx} + f_{yy} + f_{zz}.\end{aligned}$$

and for any vector field $\mathbf{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ we define a vector field curl$(\mathbf{F})$ and a scalar field div$(\mathbf{F})$ by

$$\begin{aligned}\text{curl}(\mathbf{F}) &= \text{“}\nabla \times \mathbf{F}\text{”} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle,\\ \text{div}(\mathbf{F}) &= \text{“}\nabla \bullet \mathbf{F}\text{”} = P_x + Q_y + R_z.\end{aligned}$$

(a) For any scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ check that curl(grad(f)) $= \langle 0,0,0\rangle$.
(b) For any vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ check that div(curl$(\mathbf{F})$) $= 0$.
(c) For any scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ check that div(grad(f)) $=$ laplacian(f).

(a): Consider any scalar field f and its gradient vector field grad$(f) = \langle f_x, f_y, f_z\rangle$. In order to compute the curl of grad(f) we will write grad$(f) = \langle P, Q, R\rangle$, so that $P = f_x$, $Q = f_y$ and $R = f_z$. Then by definition of the curl we have

$$\begin{aligned}\text{curl}(\text{grad}(f)) &= \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle,\\ &= \langle (f_z)_y - (f_y)_z, (f_x)_z - (f_z)_x, (f_y)_x - (f_x)_y\rangle,\\ &= \langle f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy}\rangle,\end{aligned}$$

which simplifies to $\langle 0,0,0\rangle$ because the mixed partial derivatives of f commute.
(b): Consider any vector field $\mathbf{F} = \langle P, Q, R\rangle$ and its curl

$$\text{curl}(\mathbf{F}) = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle.$$

Then by definition the divergence of curl$(\mathbf{F})$ is

$$\begin{aligned}\text{div}(\text{curl}(\mathbf{F})) &= (R_y - Q_z)_x + (P_z - R_x)_y + (Q_x - P_y)_z\\ &= (R_y)_x - (Q_z)_x + (P_z)_y - (R_x)_y + (Q_x)_z - (P_y)_z\\ &= R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz}\\ &= (P_{zy} - P_{yz}) + (Q_{xz} - Q_{zx}) + (R_{yx} - R_{xy}),\end{aligned}$$

which simplifies to $0 + 0 + 0 = 0$ because the mixed partial derivatives of P, Q, R commute.

Remark: These are the three most basic “derivative identities” of vector calulus. They are important for such topics as fluid dynamics and electro-magnetism.

(c): Consider any scalar field f and its gradient vector field grad$(f) = \langle f_x, f_y, f_z\rangle$. Then from the definition of divergence and laplacian we have

$$\text{div}(\text{grad}(f)) = (f_x)_x + (f_y)_y + (f_z)_z = f_{xx} + f_{yy} + f_{zz} = \text{laplacian}(f).$$

2. Conservative Vector Fields. Consider the vector field

$$\mathbf{F}(x,y,z) = \langle y+z, x+z, x+y\rangle\,.$$

(a) Check that the curl is constantly zero: $\nabla \times \mathbf{F}(x,y,z) = \langle 0,0,0\rangle$.

(b) It follows from part (a) that there exists some scalar field $f(x,y,z)$ satisfying $\nabla f = \mathbf{F}$. Find one such scalar field. [Hint: Integrate $\mathbf{F}$ along an arbitrary path starting at some arbitrary point and ending at (x,y,z).]

(a): Let $\mathbf{F} = \langle y+z, x+z, x+y\rangle = \langle P, Q, R\rangle$. Then we have

$$\begin{aligned}\nabla \times \mathbf{F} &= \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle \\ &= \langle (x+y)_y - (x+z)_z, (y+z)_z - (x+y)_x, (x+z)_x - (y+z)_y\rangle \\ &= \langle 1-1, 1-1, 1-1\rangle \\ &= \langle 0,0,0\rangle.\end{aligned}$$

Remark: If $\mathbf{F} = \nabla f$ for some scalar field f then we know from Problem 1(a) that $\nabla \times \mathbf{F} = \langle 0,0,0\rangle$. The "fundamental theorem of conservative vector fields" says that the converse is also true. That is, if $\nabla \times \mathbf{F} = \langle 0,0,0\rangle$ (and if $\mathbf{F}$ is continuous everywhere in $\mathbb{R}^3$, as it is in this example) then there must exist some scalar field f such that $\mathbf{F} = \nabla f$.

(b): In order to find such a scalar field we can just take $f = \int \mathbf{F} \bullet \mathbf{T}\, ds$, where the line integral is computed over **any path with endpoint** (x,y,z).[1] The easiest imaginable such path is $\mathbf{r}(t) = \langle xt, yt, zt\rangle$ for $0 \le t \le 1$. Thus we define

$$\begin{aligned}f(x,y,z) &= \int_0^1 \mathbf{F}(xt,yt,zt) \bullet \langle xt,yt,zt\rangle'\, dt \\ &= \int_0^1 \langle yt+zt, xt+zt, xt+yt\rangle \bullet \langle x,y,z\rangle\, dt \\ &= \int_0^1 [(yt+zt)x + (xt+zt)y + (xt+yt)z]\, dt \\ &= \int_0^1 [xyt + xzt + xyt + zyt + xzt + yzt]\, dt \\ &= 2(xy+xz+yz) \cdot \int_0^1 t\, dt \\ &= 2(xy+xz+yz) \cdot \left[\frac{1^2}{2} - \frac{0^2}{2}\right] \\ &= xy + xz + yz.\end{aligned}$$

We verify that this satisfies the desired property $\nabla f = \langle y+z, x+z, x+y\rangle$:

$$\begin{aligned}\nabla(xy+xz+yz) &= \langle (xy+xz+yz)_x, (xy+xz+yz)_y, (xy+xz+yz)_z\rangle \\ &= \langle y+z+0, x+0+z, 0+x+y\rangle \\ &= \langle y+z, x+z, x+y\rangle.\end{aligned}$$

3. Gravitational Potential. A sun of mass M sits at the origin in $\mathbb{R}^3$. According to Newton, the gravitational force due to the sun acting on a particle of mass m at the point (x,y,z) has the form

$$\mathbf{F}(x,y,z) = \frac{-GMm}{(x^2+y^2+z^2)^{3/2}} \cdot \langle x,y,z\rangle,$$

where G is the gravitational constant.

[1] This is the three-dimensional version of the Calculus I theorem that $g(x) = \int_a^x G(t)\, dt$ satisfies $g'(x) = G(x)$ for **any lower endpoint** a.

(a) Check that the following scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ satisfies $\nabla f = \mathbf{F}$:

$$f(x, y, z) = \frac{+GMm}{\sqrt{x^2 + y^2 + z^2}}.$$

(b) We can think of $U = -f$ as the *gravitational potential energy*. Suppose that our particle is held at rest at a position with distance D from the origin. At time zero the particle is allowed to fall towards the sun. If the sun has radius R, use conservation of energy to compute the particle's speed when it hits the sun's surface. Your answer will involve the constants G, M, R and D (but not m). [Assume that $D > R$.]

(a): First we compute the partial derivative of f with respect to x:

$$\begin{aligned} f_x &= \frac{\partial}{\partial x} GMm(x^2 + y^2 + z^2)^{-1/2} \\ &= GMm \cdot \frac{-1}{2}(x^2 + y^2 + z^2)^{-3/2}(2x + 0 + 0) \\ &= -GMmx/(x^2 + y^2 + z^2)^{3/2}. \end{aligned}$$

Identical computations show that $f_y = -GMmy/(x^2 + y^2 + z^2)^{3/2}$ and $f_z = -GMmz/(x^2 + y^2 + z^2)^{3/2}$, hence

$$\begin{aligned} \nabla f &= \langle f_x, f_y, f_z \rangle \\ &= \left\langle \frac{-GMmx}{(x^2 + y^2 + z^2)^{3/2}}, \frac{-GMmy}{(x^2 + y^2 + z^2)^{3/2}}, \frac{-GMmz}{(x^2 + y^2 + z^2)^{3/2}} \right\rangle \\ &= \frac{-GMm}{(x^2 + y^2 + z^2)^{3/2}} \cdot \langle x, y, z \rangle, \end{aligned}$$

as desired.

(b): If a particle with mass m and trajectory $\mathbf{r}(t)$ moves through the force field $\mathbf{F} = \nabla f$ then the theorem on conservation of energy says that the total mechanical energy is constant:

$$\text{KE} + \text{PE} = \frac{1}{2} m \|\mathbf{r}'(t)\|^2 - f(\mathbf{r}(t)) = \text{constant}.$$

Note that $f(\mathbf{r}(t)) = GMm/\sqrt{x(t)^2 + y(t)^2 + z(t)^2} = GMm/\|\mathbf{r}(t)\|$, where $\|\mathbf{r}(t)\|$ is just the distance from the position $\mathbf{r}(t)$ to the origin. Since the particle starts at rest with distance D to the origin we have

$$(\text{KE} + \text{PE})(\text{start}) = 0 - GMm/D.$$

Let v be the particle's speed when it hits the surface of the sun, i.e., when $\|r(t)\| = R$. At this moment we have

$$(\text{KE} + \text{PE})(\text{hit}) = \frac{1}{2} mv^2 - GMm/R.$$

Finally, conservation of energy says that

$$\begin{aligned} (\text{KE} + \text{PE})(\text{hit}) &= (\text{KE} + \text{PE})(\text{start}) \\ \frac{1}{2} mv^2 - GMm/R &= 0 - GMm/D \\ v^2 &= 2GM/R - 2GM/D \\ v &= \sqrt{\frac{2GM}{R} - \frac{2GM}{D}}. \end{aligned}$$

Note that the mass m dropped out of the equation.

Remark: This equation works well for NASA, but it breaks down in extreme cases. For example, it predicts that $v \to \infty$ for a sun with large mass M and tiny radius R.

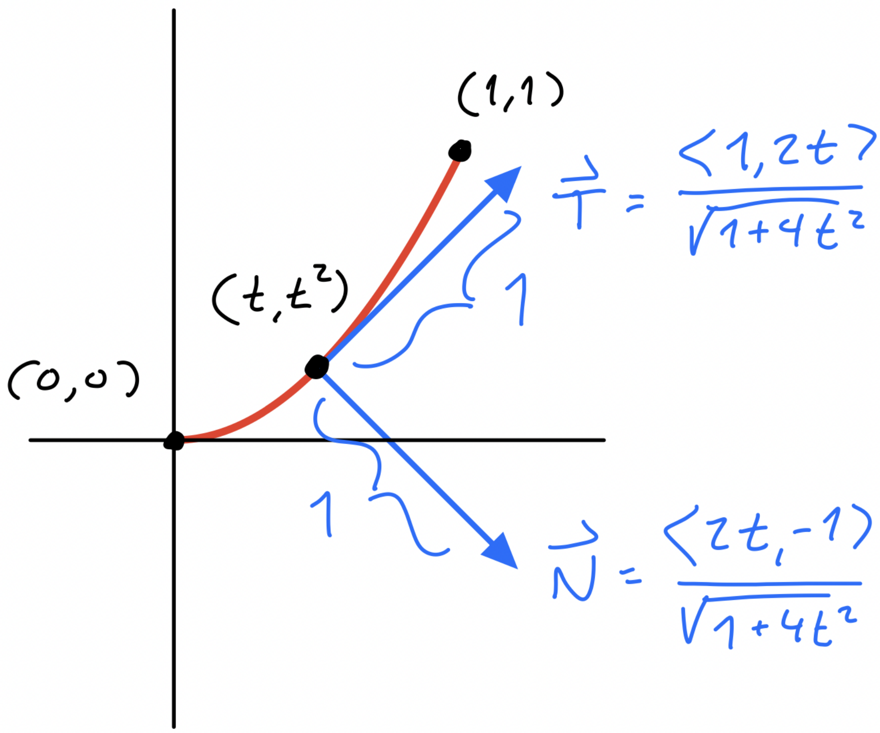
4. Line Integrals and Flux Integrals in $\mathbb{R}^2$. Let C be the parametrized path $\mathbf{r}(t) = (t, t^2)$ for $0 \leq t \leq 1$, with velocity vector $\mathbf{r}'(t) = \langle 1, 2t \rangle$. From this parametrization we can define a unit tangent vector and a unit normal vector to the curve C at the point $\mathbf{r}(t)$:

$$\mathbf{T} = \frac{\langle 1, 2t \rangle}{\sqrt{1 + 4t^2}} \quad \text{and} \quad \mathbf{N} = \frac{\langle 2t, -1 \rangle}{\sqrt{1 + 4t^2}}.$$

Now consider the constant vector field $\mathbf{F}(x, y) = \langle 3, 1 \rangle$.

(a) Compute the line integral $\int_C \mathbf{F} \bullet \mathbf{T}\, ds$.
(b) Compute the flux integral $\int_C \mathbf{F} \bullet \mathbf{N}\, ds$.

Here is a picture of the path $\mathbf{r}(t) = (t, t^2)$ for $0 \leq t \leq 1$ together with the unit tangent and normal vectors at the point (t, t^2):



Note that $\mathbf{T} = \mathbf{r}(t)/\|\mathbf{r}'(t)\|$ is just velocity vector $\mathbf{r}'(t) = \langle 1, 2t \rangle$ divided by its magnitude $\|\mathbf{r}'(t)\| = \sqrt{1^2 + (2t)^2}$. The normal vector $\mathbf{N}$ is just $\mathbf{T}$ rotated $90°$ clockwise. We do this so that $\mathbf{N}$ points "to the right" of the curve. When $\mathbf{r}(t)$ is the oriented boundary curve of a 2D region D this ensures that $\mathbf{N}$ points "out of" the region. (The region D is always "to the left" of its boundary curve ∂D.)

(a): The line integral is

$$\begin{aligned}\int \mathbf{F} \bullet \mathbf{T}\, ds &= \int_0^1 \mathbf{F}(t, t^2) \bullet \frac{\langle 1, 2t \rangle}{\|\mathbf{r}'(t)\|} \|\mathbf{r}'(t)\|\, dt \\ &= \int_0^1 \mathbf{F}(t, t^2) \bullet \langle 1, 2t \rangle\, dt\end{aligned}$$

$$= \int_0^1 \langle 3,1\rangle \bullet \langle 1,2t\rangle \, dt$$
$$= \int_0^1 (3+2t)\, dt$$
$$= \left[3t + 2\frac{t^2}{2}\right]_0^1$$
$$= 3+1$$
$$= 4.$$

(a): The flux integral is

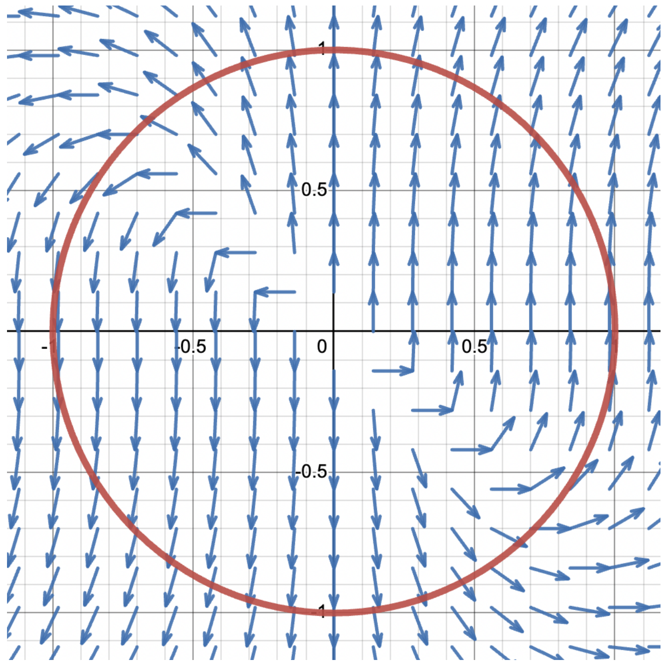
$$\int \mathbf{F}\bullet\mathbf{N}\, ds = \int_0^1 \mathbf{F}(t,t^2)\bullet \frac{\langle 2t,-1\rangle}{\|\mathbf{r}'(t)\|}\,\|\mathbf{r}'(t)\|\, dt$$
$$= \int_0^1 \mathbf{F}(t,t^2)\bullet\langle 2t,-1\rangle\, dt$$
$$= \int_0^1 \langle 3,1\rangle\bullet\langle 2t,-1\rangle\, dt$$
$$= \int_0^1 (6t-1)\, dt$$
$$= \left[6\frac{t^2}{2} - t\right]_0^1$$
$$= 3-1$$
$$= 2.$$

Remark: If we are flying a plane along the trajectory $\mathbf{r}(t)$ in a constant wind $\langle 3,1\rangle$, then the wind contributes 4 units of energy to our forward motion and 5/2 units of energy trying to push us to the right.

5. Green's Theorem on a Circle. Let D be the unit disk in $\mathbb{R}^2$ centered at $(0,0)$. Consider the vector field $\mathbf{F} = \langle P,Q\rangle = \langle xy^2, x+y\rangle$ with $\text{curl}(\mathbf{F}) = Q_x - P_y = 1-2xy$.

(a) Compute the integral $\iint_D \text{curl}(\mathbf{F})\, dA$. [Hint: Polar coordinates are easiest. You may use the trigonometric identity $\sin(2\theta) = 2\sin\theta\cos\theta$.]
(b) The boundary curve ∂D is the unit circle, oriented counterclockwise. Use the standard parametrization $\mathbf{r}(t) = (\cos t, \sin t)$ with $0 \le t \le 2\pi$ to set up the integral $\oint_{\partial D} \mathbf{F}\bullet\mathbf{T}\, ds$. You will probably not be able to evaluate this integral by hand. Use a computer to verify that you get the same answer as in part (a).

Here is a picture of the vector field $\mathbf{F} = \langle xy^2, x+y\rangle$ and the unit circle:



(a): First we integrate the scalar[2] curl($\mathbf{F}$) $= Q_x - P_y = 1 - 2xy$ over the interior of the circle. We use polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ with $dA = dxdy = r\,drd\theta$ to get

$$\begin{aligned}
\iint_D \operatorname{curl}(\mathbf{F})\,dA &= \iint_D (1 + 2xy)\,dxdy \\
&= \iint_D (1 + 2r\cos\theta r\sin\theta) r\,drd\theta \\
&= \iint_D \left[r + r^2\sin(2\theta)\right]\,drd\theta && 2\cos\theta\sin\theta = \sin(2\theta) \\
&= \int_0^{2\pi}\left(\int_0^1 \left[r + r^2\sin(2\theta)\right]\,dr\right)\,d\theta \\
&= \int_0^{2\pi}\left[\frac{r^2}{2} + \frac{r^3}{3}\cdot\sin(2\theta)\right]_0^1\,d\theta \\
&= \int_{\theta=0}^{\theta=2\pi}\left[\frac{1}{2} + \frac{1}{3}\sin(2\theta)\right]\,d\theta \\
&= \int_{u=0}^{u=4\pi}\left[\frac{1}{2} + \frac{1}{3}\sin u\right]\,\frac{du}{2} && u = 2\theta, du = 2d\theta \\
&= \frac{1}{2}\cdot\left[\frac{1}{2}\cdot u - \frac{1}{3}\cos u\right]_0^{4\pi} \\
&= \frac{1}{2}\cdot\left[\frac{1}{2}\cdot 4\pi - 0 - \frac{1}{3} + \frac{1}{3}\right] \\
&= \pi.
\end{aligned}$$

[2]The curl in 3D is a vector field. The curl in 2D is just a scalar field.

(b): Now we integrate the field $\mathbf{F} = \langle xy^2, x+y\rangle$ along the boundary curve $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$ for $0 \le t \le 2\pi$. Stokes Theorem says that we must get π as in part (a):

$$\begin{aligned}
\int_{\partial D} \mathbf{F} \bullet \mathbf{T}\, ds &= \int_0^2 \mathbf{F}(\mathbf{r}(t)) \bullet \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \cdot \|\mathbf{r}'(t)\|\, dt \\
&= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt \\
&= \int_0^{2\pi} \mathbf{F}(\cos t, \sin t) \bullet \langle -\sin t, \cos t\rangle\, dt \\
&= \int_0^{2\pi} \langle \cos t \sin^2 t, \cos t + \sin t\rangle \bullet \langle -\sin t, \cos t\rangle\, dt \\
&= \int_0^{2\pi} \left[(\cos t \sin^2 t)(-\sin t) + (\cos t + \sin t)(\cos t)\right]\, dt \\
&= \int_0^{2\pi} \left[-\cos t \sin^2 t + \cos^2 t + \cos t \sin t\right]\, dt \\
&\;\vdots \\
&= \pi.
\end{aligned}$$

This can be solved using integration by parts and various trig identities, but I used a computer.

6. Stokes' Theorem on a Parabolic Dome. Let D be the two-dimensional surface in $\mathbb{R}^3$ defined by $z = 1 - x^2 - y^2$ and $z \ge 0$. This surface can be parametrized by

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, 1 - u^2\rangle \quad \text{with } 0 \le u \le 1 \text{ and } 0 \le v \le 2\pi.$$

The boundary curve ∂D is the unit circle in the x, y-plane, oriented counterclockwise, which can be parametrized as $\mathbf{r}(t) = \langle \cos t, \sin t, 0\rangle$ for $0 \le t \le 2\pi$. Consider the vector field $\mathbf{F}(x, y, z) = \langle z, x, y\rangle$, which has constant curl vector $\nabla \times \mathbf{F} = \langle 1, 1, 1\rangle$.

(a) Compute the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ and their cross product $\mathbf{r}_u \times \mathbf{r}_v$.
(b) Use part (a) to compute the flux of the vector field $\nabla \times \mathbf{F}$ across the surface D:

$$\iint_D (\nabla \times \mathbf{F}) \bullet \mathbf{N}\, dA = \iint_D (\nabla \times \mathbf{F})(\mathbf{r}(u, v)) \bullet (\mathbf{r}_u(u, v) \times \mathbf{r}_v(u, v))\, dudv.$$

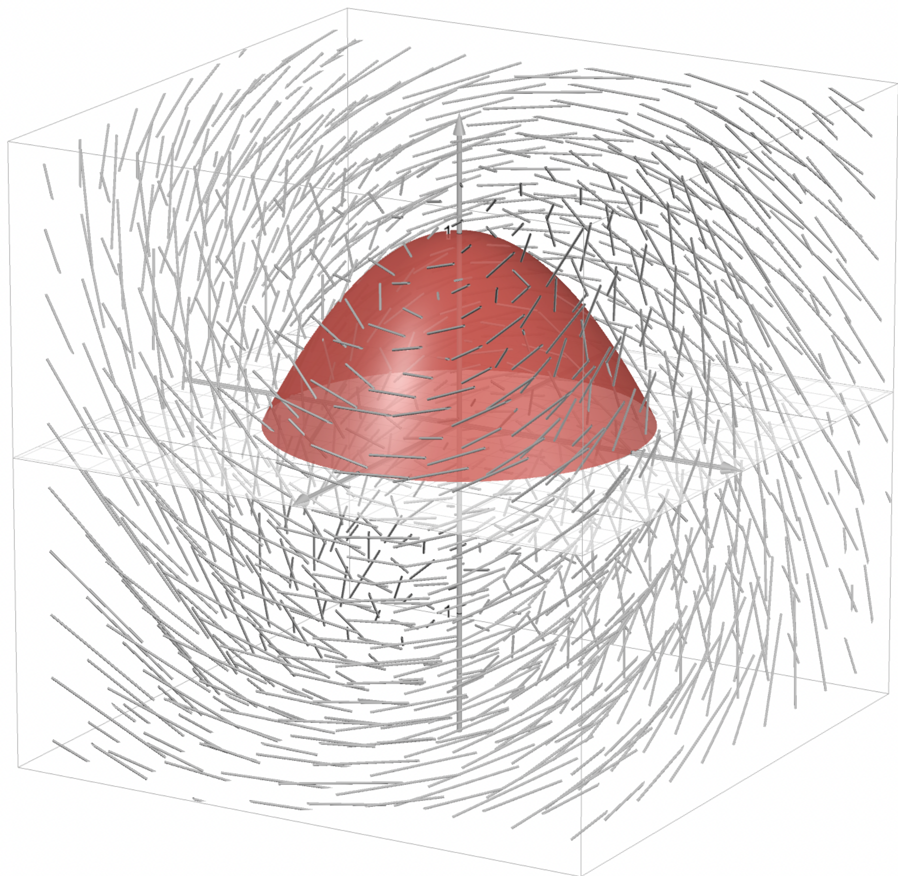
(c) Now compute the circulation of the vector field $\mathbf{F}$ around the boundary curve ∂D:

$$\int_{\partial D} \mathbf{F} \bullet \mathbf{T}\, ds = \int_{\partial D} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt.$$

Make sure that you get the same answer as in part (a).

Here is a picture[3] of the surface $\mathbf{r}(u, v)$ and the vector field $\mathbf{F}$. You can definitely see that this field has nonzero curl:

[3]You can rotate the image and play with parameters here: `https://www.desmos.com/3d/b743c9370d`



(a): First we compute the tangent vectors:

$$\mathbf{r}_u = \langle \cos v, \sin v, -2u \rangle,$$
$$\mathbf{r}_v = \langle -u \sin v, u \cos v, 0 \rangle.$$

Then we compute the cross product, which gives us the "positively oriented" normal vector:

$$\begin{aligned}\mathbf{r}_u \times \mathbf{r}_v &= \langle 2u^2 \cos v, 2u^2 \sin v, u \cos^2 v + u \sin^2 v \rangle \\ &= \langle 2u^2 \cos v, 2u^2 \sin v, u \rangle.\end{aligned}$$

The unit normal vector is $\mathbf{N} = (\mathbf{r}_u \times \mathbf{r}_v)/\|\mathbf{r}_u \times \mathbf{r}_v\|$.

(b): The flux of the constant vector field $\nabla \times \mathbf{F} = \langle 1, 1, 1 \rangle$ across the surface $\mathbf{r}(u, v)$ is

$$\begin{aligned}&\iint_D (\nabla \times \mathbf{F}) \bullet \mathbf{N}\, dA \\ &= \iint_D (\nabla \times \mathbf{F})(\mathbf{r}(u,v)) \bullet \frac{\mathbf{r}_u(u,v) \times \mathbf{r}_v(u,v)}{\|\mathbf{r}_u \times \mathbf{r}_v\|} \cdot \|\mathbf{r}_u \times \mathbf{r}_v\|\, dudv \\ &= \iint_D (\nabla \times \mathbf{F})(\mathbf{r}(u,v)) \bullet (\mathbf{r}_u(u,v) \times \mathbf{r}_v(u,v))\, dudv \\ &= \iint_D \langle 1, 1, 1 \rangle \bullet \langle 2u^2 \cos v, 2u^2 \sin v, u \rangle\, dudv \\ &= \iint_D \left(2u^2 \cos v + 2u^2 \sin v + u \right)\, dudv\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{2\pi} \left(\int_0^1 \left[2u^2 \cos v + 2u^2 \sin v + u \right] du \right) dv \\
&= \int_0^{2\pi} \left[2\frac{u^3}{3} \cos v + 2\frac{u^3}{3} \sin v + \frac{u^2}{2} \right]_0^1 dv \\
&= \int_0^{2\pi} \left[\frac{2}{3} \cos v + \frac{2}{3} \sin v + \frac{1}{2} \right] dv \\
&= \left[\frac{2}{3} \sin v - \frac{2}{3} \cos v + \frac{v}{2} \right]_0^{2\pi} \\
&= \left[0 - \frac{2}{3} + \frac{2\pi}{2} - 0 + \frac{2}{3} - 0 \right] \\
&= \pi.
\end{aligned}
$$

(c): The circulation of $\mathbf{F} = \langle z, x, y \rangle$ around the curve $\mathbf{r}(t) = (\cos t, \sin t, 0)$ for $0 \leq t \leq 2\pi$ is

$$
\begin{aligned}
\int_{\partial D} \mathbf{F} \bullet \mathbf{T} \, ds &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \bullet \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \cdot \|\mathbf{r}'(t)\| \, dt \\
&= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t) \, dt \\
&= \int_0^{2\pi} \mathbf{F}(\cos t, \sin t, 0) \bullet \langle -\sin t, \cos t, 0 \rangle \, dt \\
&= \int_0^{2\pi} \langle 0, \cos t, \sin t \rangle \bullet \langle -\sin t, \cos t, 0 \rangle \, dt \\
&= \int_0^{2\pi} \left(0 + \cos^2 t + 0 \right) dt \\
&= \int_0^{2\pi} \left(\frac{1}{2} \cos(2t) + \frac{1}{2} \right) dt \qquad \cos(2t) = 2\cos^2 t - 1 \\
&= \left[\frac{1}{4} \cdot \sin(2t) + \frac{1}{2} \cdot t \right]_0^{2\pi} \\
&= \pi.
\end{aligned}
$$

Stokes' Theorem says that the answers to (a) and (b) must be the same, and they are.