**Math 310** **Final Exam**
**Fall 2023** **Wed Dec 13**

No electronic devices are allowed. No collaboration is allowed. There are 10 pages and each page is worth 6 points, for a total of 60 points.

**1. A Plane in Space.** Consider three points in space

$$P = (1,0,0), \quad Q = (2,1,1), \quad R = (0,2,1).$$

(a) Find the equation of the plane that passes through $P$, $Q$ and $R$. [Hint: The fastest way is to compute a cross product.]

Let's define the vectors $\mathbf{u} = Q - P = \langle 1,1,1\rangle$ and $\mathbf{v} = R - P = \langle -1,2,1\rangle$. Now we compute the cross product:

$$\begin{aligned}
\mathbf{u}\times\mathbf{v} &= \text{“}\det\begin{pmatrix} \mathbf{i} & \mathbf{i} & \mathbf{k} \\ 1 & 1 & 1 \\ -1 & 2 & 1 \end{pmatrix}\text{”} \\
&= \mathbf{i}\det\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} - \mathbf{j}\det\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} + \mathbf{k}\det\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \\
&= \mathbf{i}(1-2) - \mathbf{j}(1+1) + \mathbf{k}(2+1) \\
&= -\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} \\
&= \langle -1,-2,3\rangle.
\end{aligned}$$

Since $\mathbf{u}$ and $\mathbf{v}$ are in the plane, the vector $\mathbf{u}\times\mathbf{v}$ is perpendicular to the plane. Picking any point in the plane, say $P = (1,0,0)$, the point-normal vector equation of the plane is

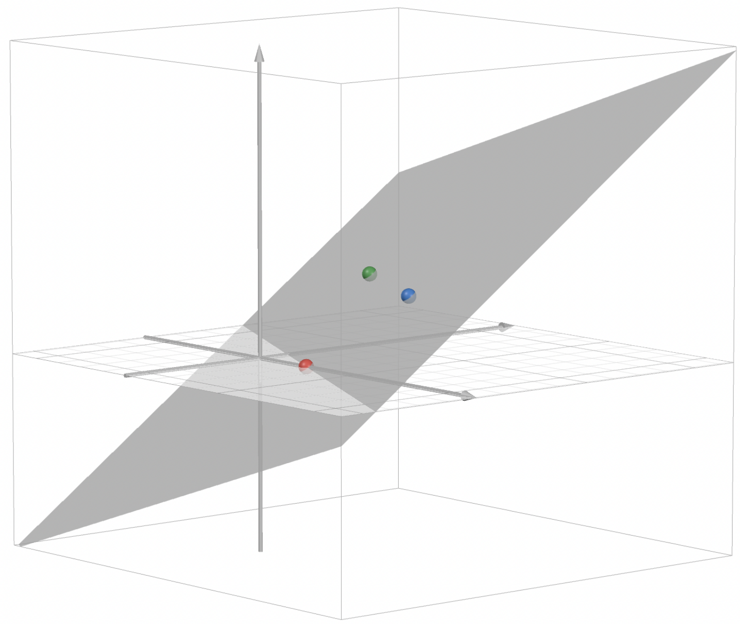
$$\begin{aligned}
\langle -1,-2,3\rangle \bullet \langle x-1, y-0, z-0\rangle &= 0 \\
-(x-1) - 2(y-0) + 3(z-0) &= 0 \\
-x - 2y + 3z + 1 &= 0 \\
x + 2y - 3z &= 1.
\end{aligned}$$

(b) Compute the **area** of the triangle $PQR$. [Hint: You can think of the triangle as half of a parallelogram.]

The area of the triangle is one half the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$. The area of the parallelogram can be computed via the dot product or the cross product. Since we already know the cross product, we find that

$$\begin{aligned}
(\text{area of triangle}) &= \frac{1}{2}(\text{area of parallelogram}) \\
&= \frac{1}{2}\|\mathbf{u}\times\mathbf{v}\| \\
&= \frac{1}{2}\sqrt{(-1)^2 + (-2)^2 + 3^2} \\
&= \frac{1}{2}\sqrt{14}.
\end{aligned}$$

Here is a picture of the plane and the three points:[1]



**2. Motion in the Plane.** Consider a path $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ with acceleration $\mathbf{r}''(t) = \langle 2, 6t \rangle$.

(a) If the initial velocity is $\mathbf{r}'(0) = \langle 0, 0 \rangle$ and the initial position is $\mathbf{r}(0) = (0, 0)$, find the position at time $t$.
Integrate once to get the velocity:

$$\begin{aligned} \mathbf{r}'(t) &= \int \mathbf{r}''(t)\, dt \\ &= \left\langle \int 2\, dt, \int 6t\, dt \right\rangle \\ &= \langle 2t + c_1, 3t^2 + c_2 \rangle \end{aligned}$$

The initial condition $\mathbf{r}'(0) = \langle 0, 0 \rangle$ gives $c_1 = 0$ and $c_2 = 0$, hence $\mathbf{r}'(t) = \langle 2t, 3t^3 \rangle$.
Integrate again to get the position:

$$\begin{aligned} \mathbf{r}(t) &= \int \mathbf{r}'(t)\, dt \\ &= \left\langle \int 2t\, dt, \int 3^2 t\, dt \right\rangle \\ &= \langle t^2 + c_3, t^3 + c_4 \rangle \end{aligned}$$

The initial condition $\mathbf{r}(0) = (0, 0)$ gives $c_3 = 0$ and $c_4 = 0$, hence $\mathbf{r}(t) = \langle t^2, t^3 \rangle$.

(b) Set up an integral to calculate the **arc length** traveled by the particle between $t = 0$ and $t = 1$. [This integral can be solved by hand but you don't need to do it.]

[1] `https://www.desmos.com/3d/69c8809e40`

The arc length traveled between $t = 0$ and $t = 1$ is

$$
\begin{aligned}
(\text{arc length}) &= \int_0^1 (\text{speed})\, dt \\
&= \int_0^1 \|\mathbf{r}'(t)\|\, dt \\
&= \int_0^1 \sqrt{(2t)^2 + (3t^2)^2}\, dt \\
&= \int_0^1 \sqrt{4t^2 + 9t^4}\, dt.
\end{aligned}
$$

We can stop here, or we can observe that there is lucky simplification:

$$
\begin{aligned}
\int_0^1 \sqrt{4t^2 + 9t^4}\, dt &= \int_0^1 \sqrt{t^2(4 + 9t^2)}\, dt \\
&= \int_0^1 t\sqrt{4 + 9t^2}\, dt && (u = 4 + 9t^2) \\
&= \int_9^{13} \frac{1}{18}\sqrt{u}\, du \\
&= \frac{1}{18}\left[\frac{2}{3}u^{3/2}\right]_4^{13} \\
&= \frac{1}{27}\left((13)^{3/2} - (4)^{3/2}\right) \\
&\approx 1.44.
\end{aligned}
$$

Here is a picture of the path:


**3. Linear Approximation.** The volume of a cone with radius $r$ and height $h$ is

$$V(r, h) = \frac{1}{3}\pi r^2 h.$$

(a) Use the chain rule to express the differential $dV$ in terms of $dr$ and $dh$.

$$\begin{aligned} dV &= \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh \\ &= \frac{2}{3}\pi r h \, dr + \frac{1}{3}\pi r^2 \, dh. \end{aligned}$$

(b) Suppose you measure the can with a ruler to find that $r = 5$ cm and $h = 9$ cm, hence $V = 75\pi$ cm$^3$. If the sensitivity of the ruler is 0.1 cm, **estimate the error** in your computed value of $V$.

The result of part (a) tells us that

$$\Delta V \approx \frac{2}{3}\pi r h \, \Delta r + \frac{1}{3}\pi r^2 \, \Delta h.$$

Substituting $r = 5$, $h = 9$ and $\Delta r = \Delta h = 0.1$ gives

$$\begin{aligned} \Delta V &\approx \frac{2}{3}\pi(5)(9)(0.1) + \frac{1}{3}\pi(5)^2(0.1) \\ &= 3\pi + 25\pi/30 \\ &= 115\pi/30. \end{aligned}$$

(The percent error is $\Delta V/V = (115\pi/30)/(75\pi) \approx 5\%$.)

**4. Tangent Plane to a Surface.** Consider the scalar field $f(x, y, z) = x^2 + y^2 + 3yz$.

(a) Compute the gradient vector field $\nabla f(x, y, z)$.

$$\begin{aligned} \nabla f &= \langle \partial f/\partial x, \partial f/\partial y, \partial f/\partial z \rangle \\ &= \langle 2x, 2y + 3z, 3y \rangle. \end{aligned}$$

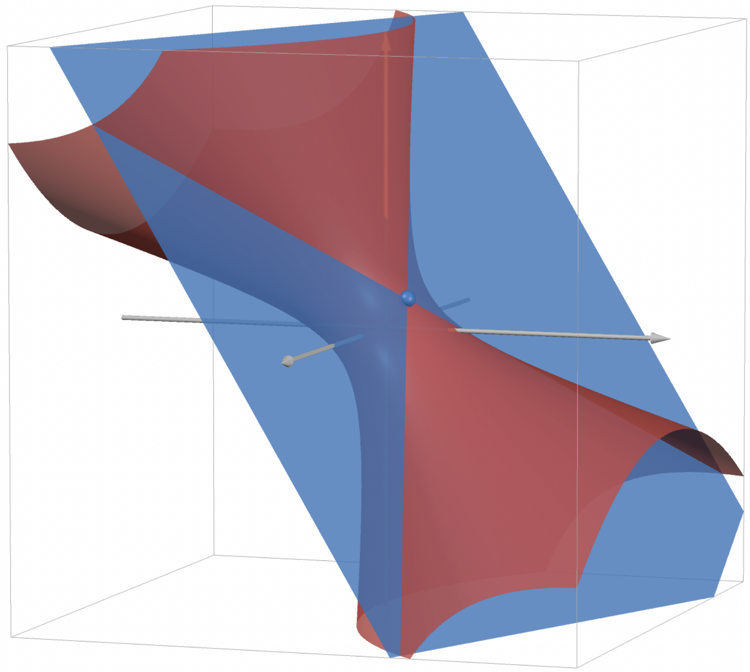
(b) Note that $f(1, 1, 1) = 1^2 + 1^2 + 3(1)(1) = 5$. Use part (a) to find the **equation of the tangent plane** to the surface $f(x, y, z) = 5$ at the point $(1, 1, 1)$.

The gradient vector $\nabla f(1, 1, 1)$ is perpendicular to the level surface $f(x, y, z) = f(1, 1, 1)$. Hence the equation of the tangent plane is

$$\begin{aligned} \nabla f(1,1,1) \bullet \langle x-1, y-1, z-1 \rangle &= 0 \\ \langle 2, 5, 3 \rangle \bullet \langle x-1, y-1, z-1 \rangle &= 0 \\ 2(x-1) + 5(y-1) + 3(z-1) &= 0 \\ 2x + 5y + 3z &= 10. \end{aligned}$$

Here is a picture:[2]

[2] `https://www.desmos.com/3d/6c14390a82`



**5. Optimization.** The scalar field $f(x, y) = x^4 - 8x^2 + y^2$ has three critical points:

$$(0, 0), \quad (-2, 0), \quad (+2, 0).$$

(a) Compute the $2 \times 2$ Hessian matrix $Hf(x, y)$ and its determinant.

We compute the first and second partial derivatives:

$$\begin{aligned} f_x &= 4x^3 - 16x \\ f_y &= 2y \\ f_{xx} &= 12x^2 - 16 \\ f_{xy} &= 0 \\ f_{yx} &= 0 \\ f_{yy} &= 2. \end{aligned}$$

Hence the Hessian matrix is

$$Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 12x^2 - 16 & 0 \\ 0 & 2 \end{pmatrix}$$

and its determinant is

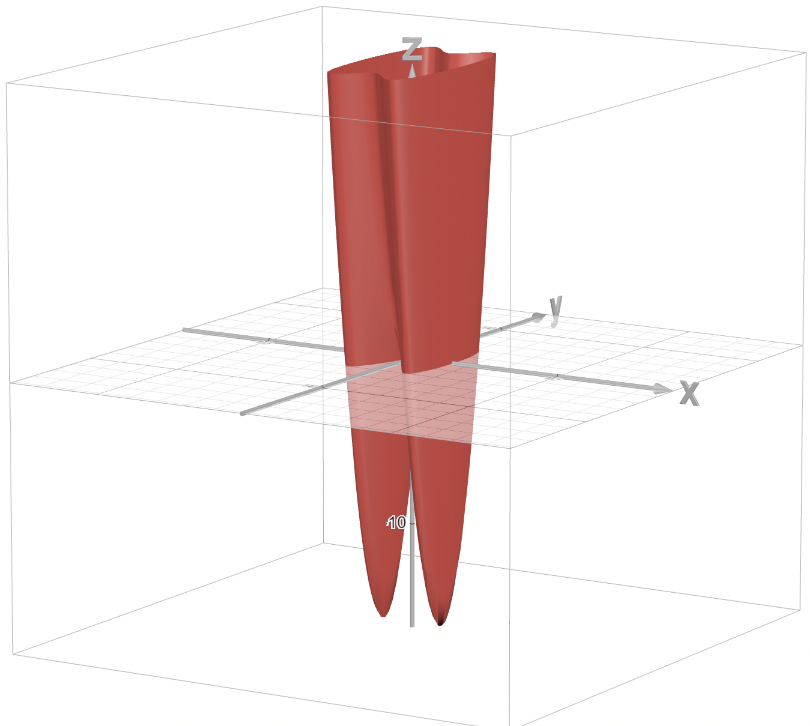
$$\det(Hf) = (12x^2 - 16)(2) - (0)(0) = 8(3x^2 - 4).$$

(b) Apply the second derivative test to determine whether each of the three critical points is a local max, a local min or a saddle point of $f$.

Since $\det(Hf)(0, 0) = 8(-4) < 0$ we conclude that $(0, 0)$ is a **saddle point** of $f$.

Since $\det(Hf)(\pm 2, 0) = 8(3(\pm 2)^2 - 4) > 0$ we conclude that each of $(\pm 2, 0)$ is a local max or min of $f$. Since $f_{yy} = 2 > 0$ at both of these points, they are both **local minima**. [We could also check that $f_{xx} > 0$ at both points. Since $\det(Hf) > 0$ the numbers $f_{xx}$ and $f_{yy}$ must have the same sign.]

Here is a picture:[3]




**6. Integration in the Plane.**

(a) Integrate $f(x, y) = x + y$ over the rectangle with $0 \le x \le 1$ and $0 \le y \le 2$.

$$\begin{aligned}\iint_{\text{rectangle}} (x+y)\,dxdy &= \int_0^2 \left( \int_0^1 (x+y)\,dx \right) dy \\ &= \int_0^2 \left[ \frac{x^2}{2} + xy \right]_0^1 dy \\ &= \int_0^2 \left( \frac{1}{2} + y \right) dy \\ &= \left[ \frac{1}{2}y + \frac{y^2}{2} \right]_0^2 \\ &= 1 + 2 \\ &= 3.\end{aligned}$$

[3] `https://www.desmos.com/3d/7788ae17e1`

(b) Integrate $f(x,y) = x + y$ over the region between the $x$-axis and the parabola $y = x^2$, for $0 \le x \le 1$. [Hint: First parametrize the region.]

This region $D$ is parametrized by $0 \le x \le 1$ and $0 \le y \le x^2$. The integral is

$$\begin{aligned}\iint_D (x+y)\,dxdy &= \int_0^1 \left( \int_0^{x^2} (x+y)\,dy \right) dx \\ &= \int_0^1 \left[ xy + \frac{y^2}{2} \right]_0^{x^2} dx \\ &= \int_0^1 \left( x^3 + \frac{x^4}{2} \right) dx \\ &= \left[ \frac{x^4}{4} y + \frac{x^5}{10} \right]_0^1 \\ &= 1/4 + 1/10 \\ &= 7/20.\end{aligned}$$

**7. Cylindrical and Spherical Coordinates.**

(a) Use cylindrical coordinates to integrate the function $f(x,y,z) = x$ over the cylinder with $x^2+y^2 \le 1$ and $0 \le z \le 1$. [Hint: We have $x = r\cos\theta$ and $dxdydz = r\,drd\theta dz$.] Let $x = r\cos\theta$ and $y = r\sin\theta$ so the cylinder is parametrized by $0 \le r \le 1$, $0 \le \theta \le 2\pi$ and $0 \le z \le 1$. The integral is

$$\begin{aligned}\iiint_{\text{cylinder}} x\,dxdydz &= \iiint_{\text{cylinder}} r\cos\theta\, r\,drd\theta dz \\ &= \int_0^1 r^2\,d\theta \cdot \int_0^{2\pi} \cos\theta\,d\theta \cdot \int_0^1 1\,dz \\ &= \left( \frac{1^3}{3} - \frac{0^3}{3} \right) (\sin(2\pi) - \sin(0))\,(1-0) \\ &= (1/3)(0)(1) \\ &= 0.\end{aligned}$$

The integral is zero because of a symmetry. Positive and negative values of $x$ cancel.

(b) Use spherical coordinates to compute the volume of the sphere $x^2 + y^2 + z^2 \le 1$. [Hint: $dxdydz = \rho^2 \sin\varphi\,d\rho d\theta d\varphi$.]

The sphere is parametrized by $0 \le \rho \le 1$, $0 \le \theta \le 2\pi$, $0 \le \varphi \le \pi$. The volume is

$$\begin{aligned}(\text{volume of sphere}) &= \iiint_{\text{sphere}} 1\,dxdydz \\ &= \iiint_{\text{sphere}} 1\,\rho^2 \sin\varphi\,d\rho d\theta d\varphi \\ &= \int_0^1 \rho^2\,d\rho \cdot \int_0^{2\pi} 1\,d\theta \cdot \int_0^{\pi} \sin\varphi\,d\varphi \\ &= \left( \frac{1^3}{3} - \frac{0^3}{3} \right) (2\pi - 0)\,(-\cos(\pi) + \cos(0))\end{aligned}$$

$$= (1/3)(2\pi)(2)$$
$$= 4\pi/3.$$

This agrees with the formula $(4/3)\pi(\text{radius})^3$ since our sphere has radius 1.

**8. Surface Area.** Consider the following parametrized surface in $\mathbb{R}^3$:

$$\mathbf{r}(u, v) = \langle 1 + u - v, u + 2v, u + v\rangle \qquad \text{with } 0 \le u \le 1 \text{ and } 0 \le v \le 1.$$

(a) Compute the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$, and the normal vector $\mathbf{r}_u \times \mathbf{r}_v$.

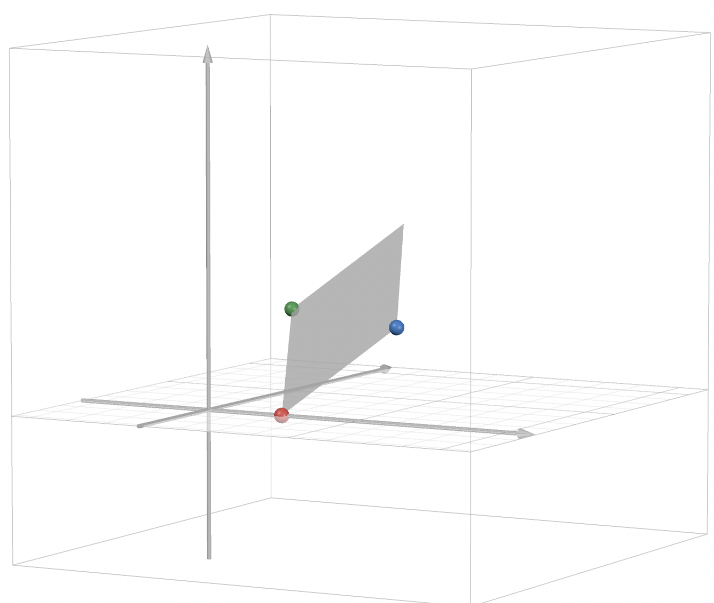
$$\begin{aligned} \mathbf{r}_u &= \langle 1, 1, 1\rangle, \\ \mathbf{r}_v &= -1, 2, 1\rangle \\ \mathbf{r}_u \times \mathbf{r}_v &= \langle -1, -2, 3\rangle. \end{aligned}$$

We already did this computation in Problem 1.

(b) Use your answer from part (a) to compute the **area of the surface**. [This surface integral is unusual because it **can** be solved by hand.]

$$\begin{aligned} (\text{surface area}) &= \iint 1 \, \|\mathbf{r}_u \times \mathbf{r}_v\| \, dudv \\ &= \iint \sqrt{(-1)^2 + (-2)^2 + 3^3} \, dudv \\ &= \int_0^1 \int_0^1 \sqrt{14} \, dudv \\ &= \sqrt{14}. \end{aligned}$$

Indeed, the surface $\mathbf{r}(u, v)$ with $0 \le u \le 1$ and $0 \le v \le 1$ is just the same parallelogram that we considered in Problem 1. Here is a picture:[4]



[4] https://www.desmos.com/3d/c1ac15e534

**9. Green's Theorem.** Consider the vector field $\mathbf{F}(x, y) = \langle P, Q\rangle = \langle y^2, xy\rangle$.

(a) Compute the line integral of $\mathbf{F}$ along the path $\mathbf{r}(t) = \langle t, t\rangle$ for $0 \le t \le 1$.

$$\begin{aligned}\int_0^1 \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt &= \int_0^1 \mathbf{F}(t, t) \bullet \langle 1, 1\rangle\, dt \\ &= \int_0^1 \langle t^2, t^2\rangle \bullet \langle 1, 1\rangle\, dt \\ &= \int_0^1 2t^2\, dt \\ &= 2/3.\end{aligned}$$

(b) Compute the line integral of $\mathbf{F}$ along the path $\mathbf{r}(t) = \langle t, t^2\rangle$ for $0 \le t \le 1$.

$$\begin{aligned}\int_0^1 \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt &= \int_0^1 \mathbf{F}(t, t^2) \bullet \langle 1, 2t\rangle\, dt \\ &= \int_0^1 \langle t^4, t^3\rangle \bullet \langle 1, 2t\rangle\, dt \\ &= \int_0^1 (t^4 + 2t^4)\, dt \\ &= \int_0^1 3t^4\, dt \\ &= 3/5.\end{aligned}$$

(c) Compute the integral of the scalar $\text{curl}(\mathbf{F}) = Q_x - P_y = -y$ over the two-dimensional region with $0 \le x \le 1$ and $x^2 \le y \le x$.

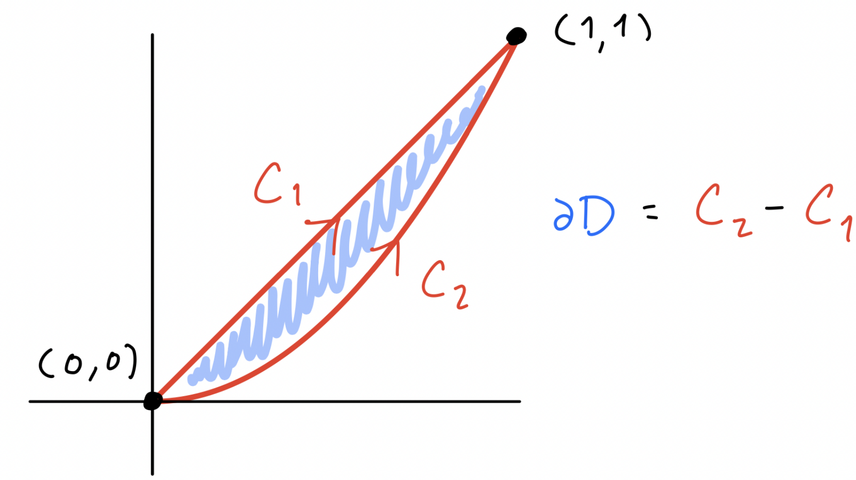
$$\begin{aligned}\iint_D \text{curl}(\mathbf{F})\, dxdy &= \iint_D (-y)\, dxdy \\ &= \int_0^1 \left(\int_{x^2}^x -y\, dy\right) dx \\ &= \int_0^1 \left[-\frac{y^2}{2}\right]_{x^2}^x dx \\ &= \int_0^1 \left(-\frac{x^2}{2} + \frac{x^4}{2}\right) dx \\ &= \left[-\frac{x^3}{6} + \frac{x^5}{10}\right]_0^1 \\ &= \left(-\frac{1}{6} + \frac{1}{10}\right) \\ &= -1/15.\end{aligned}$$

(d) Your answers to parts (a), (b) and (c) are related by Green's Theorem. Explain the relationship. It may be helpful to draw a picture.

For any two-dimensional region $D$, Green's Theorem says that

$$\iint_D \text{curl}(\mathbf{F}) = \int_{\partial D} \mathbf{F},$$

where $\partial D$ is the boundary curve of $D$, oriented so that $D$ is "to the left". In our case the region $D$ is the one described in part (c). We can describe the boundary as $\partial D = C_2 - C_1$ where $C_2$ is the path in (b) and $C_1$ is the path in (a). Picture:




It follows from Green's Theorem that

$$\iint_D \text{curl}(\mathbf{F}) = \int_{C_2 - C_1} \mathbf{F}$$

$$\iint_D \text{curl}(\mathbf{F}) = \int_{C_2} \mathbf{F} - \int_{C_1} \mathbf{F}$$

$$\text{(answer to (c))} = \text{(answer to (b))} - \text{(answer to (a))}.$$

Indeed, we verify that $-1/15 = 3/5 - 2/3$.