Math 310 **Exam 2**
Fall 2023 **Fri Dec 1**

No electronic devices are allowed. No collaboration is allowed. There are 5 pages and each page is worth 6 points, for a total of 30 points.

1. Integrating a Scalar Over a Rectangle.

(a) Integrate $f(x,y) = x + y$ over the rectangle with $-1 \le x \le 2$ and $1 \le y \le 3$.

$$\begin{aligned}
\iint f(x,y)\,dxdy &= \int_{y=1}^{3} \left(\int_{x=-1}^{2} (x+y)\,dx \right) dy \\
&= \int_{y=1}^{3} \left[\frac{x^2}{2} + xy \right]_{x=-1}^{2} dy \\
&= \int_{y=1}^{3} \left[\frac{4}{2} + 2y - \left(\frac{1}{2} - y \right) \right] dy \\
&= \int_{y=1}^{3} \left[\frac{3}{2} + 3y \right] dy \\
&= \left[\frac{3}{2}y + 3\frac{y^2}{2} \right]_{y=1}^{3} \\
&= \frac{9}{2} + \frac{27}{2} - \left(\frac{3}{2} + \frac{3}{2} \right) \\
&= 15.
\end{aligned}$$

Remark: We could view this as the volume of the region above the rectangle in the x,y-plane with $-1 \le x \le 2$ and $1 \le y \le 3$ and below the surface $z = x + y$.

(b) Compute the volume of the 3D region above the square in the x,y-plane with $0 \le x \le 1$ and $0 \le y \le 1$, and below the surface $z = x^2 y$.

We can view $x^2 y\,dxdy$ as the volume of a skinny column above the point $(x,y,0)$, where $x^2 y$ is the height of the column and $dxdy$ are the area of the base. Hence

$$\begin{aligned}
\text{Volume} &= \iint (\text{skinny columns}) \\
&= \iint x^2 y\,dxdy \\
&= \int_0^1 x^2\,dx \cdot \int_0^1 y\,dy \\
&= \left(\frac{1^3}{3} - \frac{0^3}{3} \right) \cdot \left(\frac{1^2}{2} - \frac{0^2}{2} \right) \\
&= 1/6.
\end{aligned}$$

Alternatively, some students parametrized the 3D region by $0 \le x \le 1$, $0 \le y \le 1$ and $0 \le z \le x^2 y$ and then computed

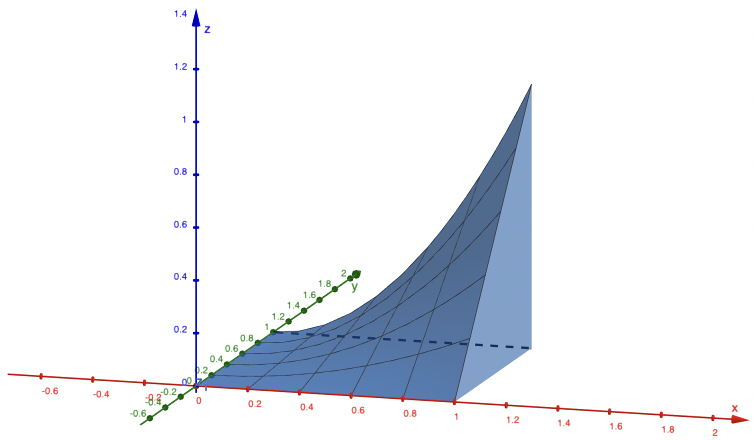
$$\text{Volume} = \iiint 1\,dxdydz$$

$$= \int_0^1 \left(\int_0^1 \left(\int_0^{x^2 y} 1 \, dz \right) dx \right) dy$$

$$= \int_0^1 \left(\int_0^1 x^2 y \, dx \right) dy$$

$$= \text{same as before.}$$

Here is a picture of the 3D region:



2. Polar and Cylindrical Coordinates.

(a) Use polar coordinates to integrate $f(x, y) = x^2 + y^2$ over the unit disk $x^2 + y^2 \leq 1$.

Let $x = r\cos\theta$ and $y = r\sin\theta$ so that $x^2 + y^2 = r^2$ and $dxdy = r\,drd\theta$. The unit disk is parametrized by $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$, so that

$$\begin{aligned}\iint_{\text{disk}} (x^2 + y^2)\, dxdy &= \iint_{\text{disk}} r^2 \cdot r\, drd\theta \\ &= \int_0^{2\pi} 1\, d\theta \cdot \int_0^1 r^3\, dr \\ &= 2\pi \cdot \left(\frac{1^4}{4} - \frac{0^4}{4} \right) \\ &= \pi/2.\end{aligned}$$

(b) Use cylindrical coordinates to integrate $f(x, y, z) = x^2 + y^2 + z^2$ over the cylinder satisfying $x^2 + y^2 \leq 1$ and $0 \leq z \leq 1$.

Let $x = r\cos\theta$ and $y = r\sin\theta$ so that $r^2 = x^2 + y^2$ and $dxdydz = r\,drd\theta dz$. The cylinder is parametrized by $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq 1$, so that

$$\begin{aligned}&\iiint_{\text{cylinder}} (x^2 + y^2 + z^2)\, dxdydz \\ &= \iiint_{\text{cylinder}} (r^2 + z^2) \cdot r\, drd\theta dz \\ &= \iiint_{\text{cylinder}} (r^3 + rz^2)\, drd\theta dz\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{2\pi} 1\, d\theta \cdot \int_{z=0}^{1} \left(\int_{r=0}^{1} (r^3 + rz^2)\, dr \right) dz \\
&= 2\pi \cdot \int_0^1 \left[\frac{r^4}{4} + \frac{r^2}{2} z^2 \right]_{r=0}^{1} dz \\
&= 2\pi \cdot \int_0^1 \left(\frac{1}{4} + \frac{1}{2} z^2 \right) dz \\
&= 2\pi \cdot \left[\frac{1}{4} z + \frac{1}{2} \cdot \frac{z^3}{3} \right]_{z=0}^{1} \\
&= 2\pi \cdot \left(\frac{1}{4} + \frac{1}{2} \cdot \frac{1}{3} \right) \\
&= 5\pi/6.
\end{aligned}
$$

3. Surface Area. Consider the following parametrized surface in 3D:

$$\mathbf{r}(u, v) = \langle u, v, u^2 + uv \rangle \qquad \text{with } 0 \le u \le 1 \text{ and } 0 \le v \le 1.$$

(a) Compute the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$, and the normal vector $\mathbf{r}_u \times \mathbf{r}_v$.

We have $\mathbf{r}_u = \langle 1, 0, 2u + v \rangle$, $\mathbf{r}_v = \langle 0, 1, u \rangle$, and

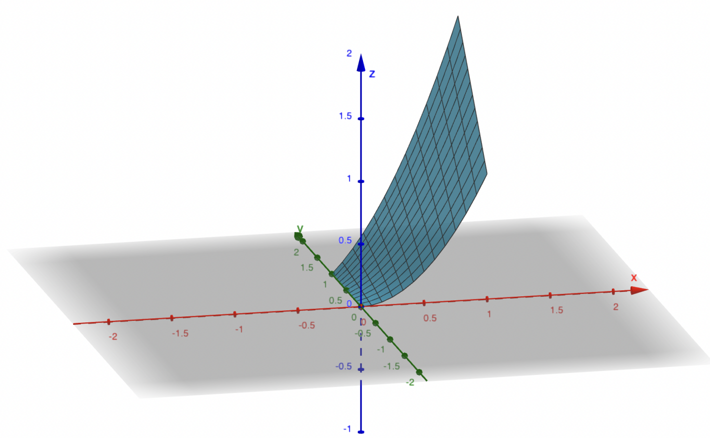
$$\mathbf{r}_u \times \mathbf{r}_v = \langle -2u - v, -u, 1 \rangle.$$

(b) Use your answer from part (a) to set up an integral to compute the **area of the surface** and simplify as much as possible. [This integral is too difficult to evaluate.]

Hence the surface area is

$$
\begin{aligned}
\iint 1 \|\mathbf{r}_u \times \mathbf{r}_v\|\, dudv &= \int_0^1 \int_0^1 \sqrt{(-2u - v)^2 + (-u)^2 + 1^2}\, dudv \\
&= \int_0^1 \int_0^1 \sqrt{5u^2 + 4uv + v^2 + 1}\, dudv.
\end{aligned}
$$

This cannot be evaluated by hand. My computer gives 1.91994. Here is a picture:



4. Green's Theorem. Consider the vector field $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle = \langle -y^3/3, x^3/3 \rangle$.

(a) Integrate the scalar $\text{curl}(\mathbf{F}) = Q_x - P_y$ over the unit disk $x^2 + y^2 \le 1$.

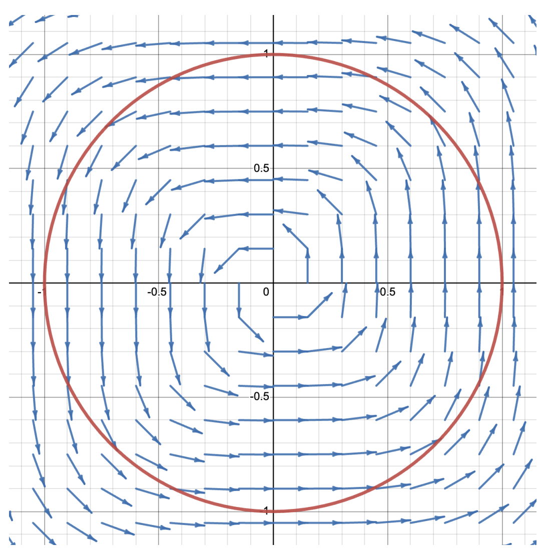
Since $Q_x - P_y = 3x^2/3 - (-3y^2/3) = x^2 + y^2$, this problem is the same as Problem 2(a). The answer is $\pi/2$.

(b) Set up the line integral of the vector field $\mathbf{F}$ around the circle $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ for $0 \le t \le 2\pi$, and simplify as much as possible. [This integral is difficult to evaluate directly, but Green's Theorem tells us that (a) and (b) have the same answer.]

The definition of the line integral gives

$$\begin{aligned}
\int \mathbf{F} \bullet \mathbf{T}\, ds &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \bullet \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \|\mathbf{r}'(t)\|\, dt \\
&= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt \\
&= \int_0^{2\pi} \mathbf{F}(\cos t, \sin t) \bullet \langle -\sin t, \cos t \rangle\, dt \\
&= \int_0^{2\pi} \langle -\sin^3 t/3, \cos^3 t/3 \rangle \bullet \langle -\sin t, \cos t \rangle\, dt \\
&= \int_0^{2\pi} \left(\frac{\sin^4 t}{3} + \frac{\cos^4 t}{3} \right) dt.
\end{aligned}$$

I guess this could be evaluated by hand, but it would take a while. The answer is $\pi/2$. Here is a picture of the disk and the vector field $\mathbf{F} = \langle -y^3/3, x^3/3 \rangle$:



5. Conservative Vector Fields. Consider the vector field $\mathbf{F}(x, y) = \langle P, Q \rangle = \langle xy^2, x^2y \rangle$. Note that this field is conservative because $Q_x = 2xy = P_y$.

(a) Find a scalar function $f(x, y)$ such that $\nabla f(x, y) = \mathbf{F}(x, y)$. [Hint: Compute the line integral of $\mathbf{F}$ along any parametrized path ending at the point (x, y).]

Let $f(x, y)$ be the line integral of $\mathbf{F}$ along the path $\langle xt, yt\rangle$ for $0 \leq t \leq 1$:

$$\begin{aligned}
f(x,y) &= \int_0^1 \mathbf{F}(xt, yt) \bullet \langle xt, yt\rangle' \, dt \\
&= \int_0^1 \langle (xt)(yt)^2, (xt)^2(yt)\rangle \bullet \langle xt, yt\rangle' \, dt \\
&= \int_0^1 \langle xy^2t^3, x^2yt^3\rangle \bullet \langle x, y\rangle \, dt \\
&= \int_0^1 \left(x^2y^2t^3 + x^2y^2t^3\right) dt \\
&= 2x^2y^2 \cdot \int_0^1 t^3 \, dt \\
&= 2x^2y^2 \cdot (1/4) \\
&= x^2y^2/2.
\end{aligned}$$

Then we check that $\nabla f = \nabla(x^2y^2/2) = \langle xy^2, x^2y\rangle = \mathbf{F}$ as desired.

(b) Use your answer from part (a) and the Fundamental Theorem of Line Integrals to compute the line integral of $\mathbf{F}$ along the path $\mathbf{r}(t) = \langle 1 + t, \sqrt{t}\rangle$ for $1 \leq t \leq 2$.

Consider the path $\mathbf{r}(t) = \langle 1 + t, \sqrt{t}\rangle$. (This is different from the path in part (a).) Since $\mathbf{F} = \nabla f$, the Fundamental Theorem of Line Integrals tells us that

$$\begin{aligned}
\int \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t) \, dt &= \int_1^2 \nabla f(\mathbf{r}(t)) \bullet \mathbf{r}'(t) \, dt \\
&= f(\mathbf{r}(2)) - f(\mathbf{r}(1)) \\
&= f(3, \sqrt{2}) - f(2, 1) \\
&= (3)^2(\sqrt{2})^2/2 - (2)^2(1)^2/2 \\
&= 9 - 2 \\
&= 7.
\end{aligned}$$

(We could also compute this without using the Fundamental Theorem, but it would take longer.) Here is a picture of the vector field $\mathbf{F} = \langle xy^2, x^2y\rangle$ and the path $\mathbf{r}$: