Math 309 **Summer 2019**
Quiz 4 **Drew Armstrong**

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1. Counting Words.

(a) Tell me the number of words of length 5 that can be made from the alphabet $\{a, b, c\}$.

$$\#(\text{words}) = \underbrace{3}_{\text{1st letter}} \times \underbrace{3}_{\text{2nd letter}} \times \underbrace{3}_{\text{3rd letter}} \times \underbrace{3}_{\text{4th letter}} \times \underbrace{3}_{\text{5th letter}} = 3^5 = 243.$$

(b) How many of the words from (a) contain 3 copies of a, 1 copy of b and 1 copy of c?

$$\binom{5}{3,1,1} = \frac{5!}{3!1!1!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 20.$$

(c) How many of the words from (a) contain 3 copies of a? [Hint: You need to add over all possible numbers of b's and c's.]

$$\binom{5}{3,2,0} + \binom{5}{3,1,1} + \binom{5}{3,0,2} = 10 + 20 + 10 = 40.$$

2. Algebraic vs Counting Proof.

For all integers $n \geq 2$ we have the following identity:

$$n^2 = 2\binom{n}{2} + n.$$

(a) Give an algebraic proof of the identity.

Proof.

$$2\binom{n}{2} + n = 2\frac{n(n-1)}{2} + n = n(n-1) + n = (n^2 - n) + n = n^2.$$

□

(b) Give a counting proof of the identity. [Hint: Count words of length 2.]

Proof. Let W be the set of words of length 2 from an alphabet of size n. On the one hand we have

$$\#W = n^2.$$

On the other hand, let $A \subseteq W$ be the words with 2 different letters and let $B \subseteq W$ be the words with the same letter twice, so $\#W = \#A + \#B$. Then we have

$$\#A = \underbrace{\binom{n}{2}}_{\text{choose two letters}} \times \underbrace{2}_{\text{put them in order}} \quad \text{and} \quad \#B = \underbrace{n}_{\text{choose one letter}}.$$

□