Math 309 **Summer 2019**
Homework 4 **Drew Armstrong**

---

1. The following table shows that the Fibonacci sequence can be run in both directions:

n	-4	-3	-2	-1	0	1	2	3	4
F_n	-3	2	-1	1	0	1	1	2	3

Use induction to prove that $F_{-n} = (-1)^{n+1} F_n$ for all $n \geq 0$.

2. For all integers $n \geq k > 0$ we have $k\binom{n}{k} = n\binom{n-1}{k-1}$.

(a) Prove this using pure algebra.
(b) Prove this using a counting argument. [Hint: Choose a committee of k people from n people. The committee has a president.]

3. Count the possibilities in each case.

(a) A phone number consists of 7 digits.
(b) Suppose that a license plate consists of 3 digits followed by 4 letters.[1]
(c) A poker hand consists of 5 unordered cards from a standard deck of 52.
(d) Solutions $x_1, x_2, x_3, x_4, x_5 \in \mathbb{N}$ to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 10$.

4. For all integers $r, g, n \geq 0$ we have the following identity:

$$\sum_{k=0}^{n} \binom{r}{k}\binom{g}{n-k} = \binom{r+g}{n}.$$

(a) Prove this identity. [Hint: There are r red balls and g green balls in an urn. You reach in and grab n balls (unordered and without repetition). Count the number of possibilities in two different ways.]
(b) Use the result of (a) to prove that $\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$.

5. The trinomial coefficient $\binom{n}{i,j,k} = \frac{n!}{i!j!k!}$ is the number of words of length n from the alphabet $\{a, b, c\}$ using i copies of a, j copies of b and k copies of c. These numbers satisfy the *trinomial recurrence*:

$$\binom{n}{i,j,k} = \binom{n-1}{i-1,j,k} + \binom{n-1}{i,j-1,k} + \binom{n-1}{i,j,k-1}.$$

(a) Prove the trinomial recurrence using pure algebra.
(b) Prove the trinomial recurrence using a counting argument.

6. Let $k \geq 0$ be an integer. Then for any number z the following formula makes sense:

$$\binom{z}{k} := \frac{1}{k!} \cdot z(z-1)(z-2)\cdots(z-k+1).$$

Isaac Newton proved that for all numbers z, x with $|x| < 1$ the following series converges:

$$(1+x)^z = \binom{z}{0} + \binom{z}{1}x + \binom{z}{2}x^2 + \binom{z}{3}x^3 + \cdots.$$

(a) For all integers $n, k \geq 0$ show that $\binom{-n}{k} = (-1)^k\binom{n+k-1}{k}$.
(b) Use part (a) to obtain the power series expansion of $(1+x)^{-2}$.

---

[1] Assume that the alphabet has 26 letters.