Math 309 **Summer 2019**
Homework 2 **Drew Armstrong**

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1. Use a truth table to verify de Morgan's laws:

$$\neg(P \wedge Q) = \neg P \vee \neg Q \qquad \text{and} \qquad \neg(P \vee Q) = \neg P \wedge \neg Q.$$

2. Compute the disjunctive normal form of the following Boolean function. Use this to draw a circuit diagram for the function.

P	Q	R	$f(P,Q,R)$
T	T	T	F
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	F
F	T	F	T
F	F	T	F
F	F	F	T

3. Let B be a Boolean algebra. For all $P, Q \in B$ we define the *Sheffer stroke* as follows:

$$P \uparrow Q := \neg(P \wedge Q).$$

Use abstract Boolean algebra to prove the following identities. Don't use truth tables!

(a) $\neg P = P \uparrow P$
(b) $P \vee Q = (P \uparrow P) \uparrow (Q \uparrow Q)$
(c) $P \wedge Q = (P \uparrow Q) \uparrow (P \uparrow Q)$

In logic the Sheffer stroke is called NAND. The formulas above demonstrate that any circuit can be built entirely from NAND gates. This is how solid state drives work.

4. Let $f : S \to T$ be a function of finite sets and for all $t \in T$ define the number

$$d(t) := \#\{s \in S : f(s) = t\}.$$

We say that f is *injective* if $d(t) \leq 1$ for all $t \in T$, *surjective* if $d(t) \geq 1$ for all $t \in T$ and *bijective* if $d(t) = 1$ for all T.

(a) If $f : S \to T$ is injective prove that $\#S \leq \#T$.
(b) If $f : S \to T$ is subjective prove tha $\#S \geq \#T$.
(c) If $f : S \to T$ is bijective prove that $\#S = \#T$.

[Hint: Observe that $\sum_{t \in T} d(t) = \#S$.]

5. Let S and T be finite sets. Explain why there are $\#T^{\#S}$ different functions from S to T.

6. (a) Explicitly write down all of the subsets of $\{1, 2, 3\}$.
(b) Explicitly write down all of the functions $\{1, 2, 3\} \to \{T, F\}$.
(c) For any finite set S describe a bijection between the subsets of S and the functions from $S \to \{T, F\}$.
(d) Combine Problems 4(c), 5 and 6(c) to count the subsets of S.

7. (a) How many functions are there from $\{1, 2, 3\}$ to $\{1, 2, 3\}$? (Don't write them down.)
(b) How many of the functions from part (a) are bijections? Write them all down.
(c) If S is a set of size n, tell me the number of bijections $S \to S$.