**Math 309** **Summer 2019**
**Homework 1** **Drew Armstrong**

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**1.** Simplify the following sum as much as possible:

$$\sum_{k=0}^{n} \frac{(k+1)(k+2)}{2} = ?$$

**2.** Find a closed formula for the sum of the first $n$ odd numbers:

$$1 + 3 + 5 + 7 + \cdots + (2n-1) = ?$$

**3.** For any integers $1 < a < b$, find a closed formula for the sum of all integers between them:

$$a + (a+1) + \cdots + (b-1) + b = ?$$

**4.** The sequence of *factorials* $0!, 1!, 2!, \ldots$ is defined by the following initial condition and recurrence relation:

$$n! := \begin{cases} 1 & \text{if } n = 0, \\ (n-1)! \cdot n & \text{if } n \geq 1. \end{cases}$$

Prove by induction that we have

$$n! > 2^n \qquad \text{for all } n \geq 4.$$

**5.** The *Fibonacci sequence* $F_0, F_1, F_2, \ldots$ is defined by the following initial conditions and recurrence relation:

$$F_n := \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ F_{n-1} + F_{n-2} & \text{if } n \geq 2. \end{cases}$$

Let $\varphi := (1 + \sqrt{5})/2$ be the *golden ratio*, which satisfies $\varphi^2 = \varphi + 1$ (check it if you don't believe me). Prove by induction that we have

$$\varphi^{n-2} < F_n < \varphi^{n-1} \qquad \text{for all } n \geq 3.$$

[Hint: Use strong induction with two base cases.]

**6.** Use induction to verify the following formula:

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4} \qquad \text{for all } n \geq 1.$$

**7.** Let $\binom{n}{k}$ be the entry in the $n$-th row and $k$-th diagonal of Pascal's triangle. The *binomial theorem* tells us that for any number $x$ and for any whole number $n \geq 0$ we have

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Use this fact to simplify the following sums as much as possible:

$$\sum_{k=0}^{n} \binom{n}{k} = ? \qquad \text{and} \qquad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = ?$$