MTH 309 **Summer 2019**
Review for Final Exam **Drew Armstrong**

Topics from Chapter 1

- Sum of consecutive integers: The following equation holds for all integers $n \geqslant 1$:

$$1+2+3+\cdots+n=\sum_{k=1}^{n} k=\frac{n(n+1)}{2}=\binom{n+1}{2}.$$

- Proof by induction:

  Base Case. The formula holds when $n=1$ because $1=1(1+1)/2$.

  Induction Step. Now fix some $n \geqslant 1$ and assume for induction that

$$1+2+\cdots+n=n(n+1)/2.$$

  In this case we also have

$$\begin{aligned} 1+2+\cdots+(n+1) &= (1+2+\cdots+n)+(n+1) \\ &= n(n+1)/2+(n+1) \\ &= (n+1)\,[n/2+1] \\ &= (n+1)(n+2)/2. \end{aligned}$$

- Principle of Induction: Let $P(n)$ be a statement depending on an integer $n \in \mathbb{Z}$. If (**Base Case**) $P(b)=T$ for some $b \in \mathbb{Z}$ and if (**Induction Step**) $P(n) \Rightarrow P(n+1)$ for all $n \geqslant b$ then we conclude that $P(n)=T$ for all $n \geqslant b$.

- Sum of consecutive squares: The following equation holds for all integers $n \geqslant 1$:

$$1^2+2^2+3^2+\cdots+n^2=\sum_{k=1}^{n} k^2=\frac{n(n+1)(2n+1)}{6}.$$

  Exercise: Prove this by induction.

- Thus, for any numbers a, b, c we have

$$\begin{aligned} \sum_{k=1}^{n}(ak^2+bk+c) &= a\left(\sum_{k=1}^{n} k^2\right)+b\left(\sum_{k=1}^{n} k\right)+c\left(\sum_{k=1}^{n} 1\right) \\ &= a\cdot\frac{n(n+1)(2n+1)}{6}+b\cdot\frac{n(n+1)}{2}+cn. \end{aligned}$$

- The Fibonacci numbers are defined by recursion:

$$F_n := \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ F_{n-1} + F_{n-2} & \text{otherwise.} \end{cases}$$

- Strong Induction: Let $P(n)$ be a statement depending on an integer $n \in \mathbb{Z}$. If (**Base Case**) $P(b) = T$ for some $b \in \mathbb{Z}$ and if (**Induction Step**)

$$[P(b) \wedge P(b+1) \wedge \cdots \wedge P(n)] \Rightarrow P(n+1) \quad \text{for all } n \geqslant b$$

then we conclude that $P(n) = T$ for all $n \geqslant b$.

- Let $\varphi = (1+\sqrt{5})/2$ and $\psi = (1-\sqrt{5})/2$ be the two roots of the equation $x^2 - x - 1 = 0$. It follows that $\alpha^2 = \alpha + 1$ and hence $\alpha^n = \alpha^{n-1} + \alpha^{n-2}$ for all n, and the same formula holds for β. Now I claim that

$$F_n = \frac{1}{\sqrt{5}} \left[\varphi^n - \psi^n\right] \quad \text{for all } n \geqslant 0.$$

Proof by Strong Induction:

Bases Cases. When $n = 0$ we have $(\varphi^0 - \psi^0)/\sqrt{5} = 0 = F_0$. When $n = 1$ we have $\varphi - \psi = \sqrt{5}$ and hence $(\varphi^1 - \psi^1)/\sqrt{5} = 1 = F_1$. That's enough.

Induction Step. Fix some $n \geqslant 0$ and assume for induction that the formula holds for all smaller values of n. Then we have

$$\begin{aligned} F_n &= F_{n-1} + F_{n-2} && \text{definition} \\ &= \frac{1}{\sqrt{5}}\left[\varphi^{n-1} - \psi^{n-1}\right] + \frac{1}{\sqrt{5}}\left[\varphi^{n-2} - \psi^{n-2}\right] && \text{induction} \\ &= \frac{1}{\sqrt{5}}\left[\varphi^{n-1} + \varphi^{n-2}\right] - \frac{1}{\sqrt{5}}\left[\psi^{n-1} + \psi^{n-2}\right] \\ &= \frac{1}{\sqrt{5}}\left[\varphi^{n}\right] - \frac{1}{\sqrt{5}}\left[\psi^{n}\right] \\ &= \frac{1}{\sqrt{5}}\left[\varphi^n - \psi^n\right]. \end{aligned}$$

- For integers $0 \leqslant k \leqslant n$ we define the entries of Pascal's triangle by recursion:

$$\binom{n}{k} := \begin{cases} 1 & k = 0 \text{ or } k = n, \\ \binom{n-1}{k-1} + \binom{n-1}{k} & 0 < k < n. \end{cases}$$

- Then one can prove the following two theorems by recursion.

Closed Formula. For all integers $0 \leqslant k \leqslant n$ we have

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Exercise: Prove this. Don't forget that $0! := 1$.

Binomial Theorem. For all numbers x and for all integers $n \geqslant 0$ we have

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

You do not need to prove this. [Here is the proof: $(1+x)^n = x(1+x)^{n-1} + (1+x)^{n-1}$.]

Topics from Chapter 2

- A set is "a collection of things," where order and repetition do not matter:

$$\{1,2,3\} = \{3,1,2\} = \{1,1,2,2,3,3,2,2,3,1,1\}.$$

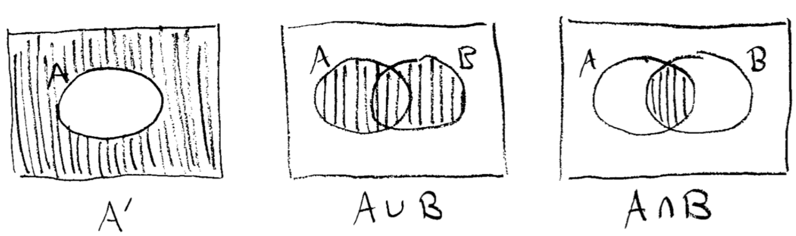
- We write $A \subseteq B$ to mean $\forall x, x \in A \Rightarrow x \in B$ and we say "A is a subset of B."
- From now on, all sets are subsets of a universal set U. Then for all $A \subseteq U$ we define

$$A' := \{x \in U : x \notin A\}$$

  and for all $A, B \subseteq U$ we define

$$A \cup B := \{x \in U : x \in A \text{ or } x \in B\},$$
$$A \cap B := \{x \in U : x \in A \text{ and } x \in B\}.$$

- The pictures are called Venn diagrams:



- The algebra of sets satisfies various algebraic identities, such as:

$$A \cup \varnothing = A,$$

$$\begin{aligned} A \cap U &= A, \\ A \cap (B \cup C) &= (A \cap B) \cup (A \cap C), \\ (A \cap B)' &= A' \cup B', \\ &\vdots \end{aligned}$$

These identities can be "proved" using Venn diagrams, but mostly they are just obvious.

- The Cartesian product of sets S and T is the set of "ordered pairs:"

$$S \times T := \{(s,t) : s \in S, t \in T\}.$$

It the sets are finite then $\#(S \times T) = \#S \times \#T$, hence the name.

- A function $f : S \to T$ from set S (called the domain) to a set T (called the codomain) is a subset of the Cartesian product: $f \subseteq S \times T$. There is only one rule: For each $s \in S$ **there exists a unique** $t \in T$ such that $(s,t) \in f$. We give this unique element t a special name:

$$\text{“}t = f(s).\text{”}$$

If S and T are finite then

$$\#\{\text{functions } S \to T\} = (\#T)^{(\#S)}.$$

- A function $f : S \to T$ is injective if $f(s_1) = f(s_2)$ implies $s_1 = s_2$. The function is surjective if for all $t \in T$ there exists some $s \in S$ such that $f(s) = t$. The function is bijective if it is both injective and surjective. Observe that

$$\begin{aligned} \exists \text{ injective } f : S \to T &\Rightarrow \#S \leqslant \#T \\ \exists \text{ surjective } f : S \to T &\Rightarrow \#S \geqslant \#T \\ \exists \text{ bijective } f : S \to T &\Rightarrow \#S = \#T \end{aligned}$$

- Example: There exists a bijection between the set of subsets of U and the set of functions $U \to \{T, F\}$, hence

$$\#\{\text{subsets of } U\} = \#\{\text{functions } U \to \{T,F\}\} = (\#\{T,F\})^{(\#U)} = 2^{(\#U)}.$$

Exercise: Describe this bijection.

- A Boolean function has the form $f : \{T,F\}^n \to \{T,F\}^m$. The number of such functions is $(2^m)^{(2^n)}$. Most of the 16 functions $f : \{T,F\}^2 \to \{T,F\}$ have special names:

P	Q	NOT P $\neg P$	P OR Q $P \vee Q$	P AND Q $P \wedge Q$	P XOR Q $P \oplus Q$	IF P THEN Q $P \Rightarrow Q$
T	T	F	T	T	F	T
T	F	F	T	F	T	F
F	T	T	T	F	T	T
F	F	T	F	F	F	T

- The algebra of sets and Boolean functions are related as follows:

$$A' = \{x \in U : \neg(x \in A)\},$$
$$A \cup B = \{x \in U : (x \in A) \vee (x \in B)\},$$
$$A \cap B = \{x \in U : (x \in A) \wedge (x \in B)\}.$$

  They satisfy all of the same algebraic identities.

- De Morgan's Laws make more sense in terms of logic. For all $x \in U$ let $P(x) \in \{T, F\}$. Then we have

$$\neg\left(\forall x \in U, P(x)\right) = \neg\left(\bigwedge_{x\in U} P(x)\right) = \left(\bigvee_{x\in U} \neg P(x)\right) = \left(\exists x \in U, \neg P(x)\right)$$

  and

$$\neg\left(\exists x \in U, P(x)\right) = \neg\left(\bigvee_{x\in U} P(x)\right) = \left(\bigwedge_{x\in U} \neg P(x)\right) = \left(\forall x \in U, \neg P(x)\right)$$

  Exercise: Translate these statements into English.

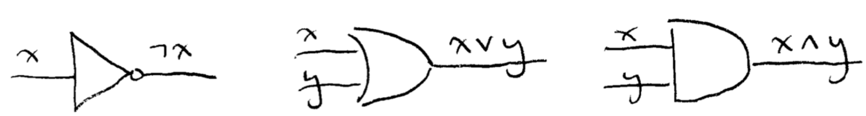
- The Principle of the Contrapositive says that $(P \Rightarrow Q) = (\neg Q \Rightarrow \neg P)$ for all $P, Q \in \{T, F\}$. We can prove it with a truth table:

P	Q	$P \Rightarrow Q$	$\neg Q$	$\neg P$	$\neg Q \Rightarrow \neg P$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

- Or we can prove it using Boolean algebra. First check that $(P \Rightarrow Q) = (\neg P \vee Q)$ for all $P, Q \in \{T, F\}$. Then we have

$$(\neg Q \Rightarrow \neg P) = (\neg(\neg Q) \vee \neg P) = (Q \vee \neg P) = (\neg P \vee Q) = (P \Rightarrow Q).$$

- We can draw many pictures of a Boolean function $f : \{T, F\}^m \to \{T, F\}^n$ by wiring together the following logic gates:



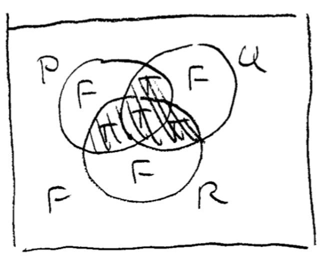
- For example, let $f : \{T, F\}^3 \to \{T, F\}$ be defined by the following table:

P	Q	R	$f(P, Q, R)$
T	T	T	$\boxed{T}$
T	T	F	$\boxed{T}$
T	F	T	$\boxed{T}$
T	F	F	F
F	T	T	$\boxed{T}$
F	T	F	F
F	F	T	F
F	F	F	F

By naming the disjunction of the T-rows we obtain the "disjunctive normal form:"

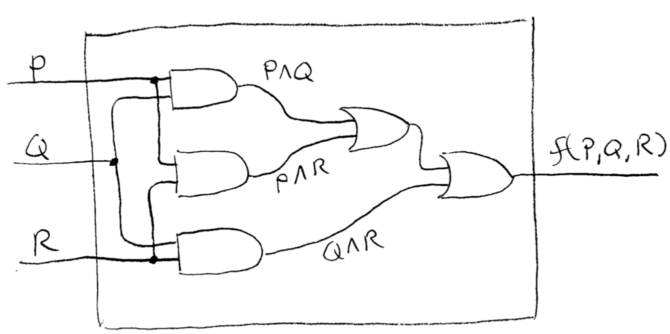
$$f(P, Q, R) = (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge R) \vee (\neg P \wedge Q \wedge R).$$

We can find a simpler expression if we draw the Venn diagram:



$$f(P, Q, R) = (P \wedge Q) \vee (P \wedge R) \vee (Q \wedge R)$$

And here is a picture of the corresponding circuit:



Topics from Chapter 3

- The integers $(\mathbb{Z}, =, <, +, \times, 0, 1)$ are defined by a bunch of obvious axioms, such as:

$$\begin{aligned} a &= a, \\ a + 0 &= a, \\ 1a &= a, \\ a + (b + c) &= (a + b) + c, \\ a(b + c) &= ab + ac, \\ &\vdots \end{aligned}$$

  together with one non-obvious axiom called induction or well-ordering.

- The Well-Ordering Principle: Any non-empty set of integers that is bounded below has a least element. In other words, if $S \subseteq \mathbb{Z}$ satisfies $S \neq \varnothing$ and if $\exists b \in \mathbb{Z}, \forall a \in S, b \leqslant a$ then $\exists m \in S, \forall a \in S, m \leqslant a$.

- Application of Well-Ordering: 1 is the least positive integer. In other words, there are no integers between 0 and 1.

  Proof: Let S be the set of positive integers, which is bounded below by 0. Since S is non-empty ($1 \in S$) **we conclude from well-ordering that S has a least element** $m \in S$. I claim that $m = 1$. Indeed, since 1 is positive and since m is the **least** positive integer we must have $m \leqslant 1$. Now assume for contradiction that $m < 1$. Then multiplying both sides of $m < 1$ by m gives $m^2 < m$ and multiplying both sides of $0 < m$ by m gives $0 < m^2$, hence m^2 is a positive integer that is smaller than m. Contradiction. We conclude that $m = 1$ and hence 1 is the least positive integer. $\square$

- Another form of Well-Ordering: There does **not** exist an infinite decreasing sequence of integers that is bounded below:

$$r_0 > r_1 > r_2 > r_3 > \cdots \geqslant b.$$

  This is the reason that algorithms terminate.

- The Division Algorithm: Given $a, b \in \mathbb{Z}$ with $a \geqslant 0$ and $b > 0$ there exist unique $q, r \in \mathbb{Z}$ such that

$$\begin{cases} a = qb + r, \\ 0 \leqslant r < b \end{cases}$$

  Proof of Existence: Keep subtracting b from a until you get a number less than b. Call it $r := a - qb < b$. We must have $r \geqslant 0$ because the number was greater than or equal to b on the second last iteration. If the algorithm went on forever we would obtain an infinite sequence:

$$a > a - b > a - 2b > a - 3b > \cdots \geqslant b.$$

  Hence the algorithm must terminate with $a = qb + r$ and $0 \leqslant r < b$. $\square$

You don't need to know the proof of uniqueness.

- First Application of Division: Base b Arithmetic. Fix some integer $b \geqslant 2$. Then for each integer $n \geqslant 0$ there exists a unique sequence $r_0, r_1, r_2, \ldots \in \{0, 1, \ldots, b-1\}$ such that

$$n = r_0 + r_1 b + r_2 b^2 + r_3 b^3 + \cdots .$$

In this case we write $n = (\cdots r_2 r_1 r_0)_b$.

Proof: Divide n by b to get $b = q_0 b + r_0$. Then continue to divide the quotient by b to get $q_{i-1} = q_i b + r_i$. The algorithm must terminate because $b > 1$ implies $q_{i-1} > q_i$. Uniqueness follows from uniqueness of remainders. $\square$

- Example: Express 101 in base 3:

$$\left\{ \begin{array}{rll} \mathbf{101} & = 3 \cdot \mathbf{33} & +2 \\ \mathbf{33} & = 3 \cdot \mathbf{11} & +0 \\ \mathbf{11} & = 3 \cdot \mathbf{3} & +2 \\ \mathbf{3} & = 3 \cdot \mathbf{1} & +0 \\ \mathbf{1} & = 3 \cdot \mathbf{0} & +1 \end{array} \right\} \implies \quad (101)_{10} = (10202)_3.$$

- Second Application of Division: Euclidean Algorithm. To compute the gcd of $a, b \in \mathbb{Z}$ with $b > 0$, first divide a by b to get $a = q_1 b + r_1$. Then divide b by r_1 to get $b = q_2 r_1 + r_2$. Continue to divide r_{i-1} by r_i to get a decreasing sequence of remainders:

$$b > r_1 > r_2 > \cdots \geqslant 0.$$

By well-ordering this must stop. The last non-zero remainder equals $\gcd(a, b)$.

Proof: If $r_{i-1} = q_{i+1} r_i + r_{i+1}$ then $\gcd(r_{i-1}, r_i) = \gcd(r_i, r_{i+1})$. More generally, if $a = xb + c$ then $\gcd(a, b) = \gcd(b, c)$. Indeed, let $d = \gcd(a, b)$ and $e = \gcd(b, c)$. Since $d|a$ and $d|b$ one can check that d divides $c = a - xb$, hence $d \leqslant e$. Conversely, since $e|b$ and $e|c$ one can check that e divides $a = xb + c$, hence $e \leqslant d$. $\square$

- Example: Compute $\gcd(101, 82)$:

$$\left\{ \begin{array}{rll} \mathbf{101} & = 1 \cdot \mathbf{82} & +\mathbf{19} \\ \mathbf{82} & = 4 \cdot \mathbf{19} & +\mathbf{6} \\ \mathbf{19} & = 3 \cdot \mathbf{6} & +\boxed{\mathbf{1}} \\ \mathbf{6} & = 6 \cdot \mathbf{1} & +\mathbf{0} \end{array} \right\} \implies \quad \gcd(101, 82) = 1.$$

Bonus: The quotients tell us that

$$\frac{101}{82} = 1 + \cfrac{1}{4 + \frac{1}{3 + \frac{1}{6}}}.$$

Topics from Chapter 4

- The Multiplication Principle: When a sequence of choices is made, the possibilities multiply. Sometimes this is drawn as a branching "decision tree."
- Words: The number of words of length k from an alphabet is size n is

$$\underbrace{n}_{\text{1st letter}} \times \underbrace{n}_{\text{2nd letter}} \times \cdots \times \underbrace{n}_{k\text{th letter}} = n^k.$$

- Permutations: The number of permutations of k things taken from n things is

$$\underbrace{n}_{\text{1st letter}} \times \underbrace{(n-1)}_{\text{2nd letter}} \times \cdots \times \underbrace{(n-(k-1))}_{k\text{th letter}} = n(n-1)\cdots(n-k+1).$$

  If $k \leqslant n$ then we can simplify this to

$$n(n-1)\cdots(n-k+1) = n(n-1)\cdots(n-k+1)\frac{(n-k)\cdots 3\cdot 2\cdot 1}{(n-k)\cdots 3\cdot 2\cdot 1} = \frac{n!}{(n-k)!}$$

- Combinations: Let ${}_nC_k$ be the number of subsets of size k from a set of size n, equivalently the number of ways to choose k unordered things without repetition from n things. Furthermore, let ${}_nP_k$ be the number of ways to choose k **ordered** things without repetition. We showed above that

$${}_nP_k = \frac{n!}{(n-k)!}.$$

  On the other hand, we can create an ordered selection by first choosing an unordered selection and then ordering it:

$${}_nP_k = \underbrace{{}_nC_k}_{\text{choose unordered selection}} \times \underbrace{k!}_{\text{then put it in order}}.$$

  It follows that

$${}_nC_k = \frac{{}_nP_k}{k!} = \frac{n!/(n-k)!}{k!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}.$$

  Was that a surprise?

- We can prove the same result by induction:

  Boundary Cases. If $k=0$ or $n=0$ then we have ${}_nC_k = 1$ because there is one way to choose nothing and one way to choose everything.

  Recursion. Let S be the set of subsets of size k from $\{1,2,\ldots,n\}$ so that $\#S = {}_nC_k$. We can break this set into two pieces:

$$S' := \{A \subseteq \{1,\ldots,n\} : \#A = k \text{ and } n \in A\},$$
$$S'' := \{A \subseteq \{1,\ldots,n\} : \#A = k \text{ and } n \notin A\}.$$

  Exercise: Explain why $\#S' = {}_{n-1}C_{k-1}$ and $\#S'' = {}_{n-1}C_k$. It follows that

$${}_nC_k = \#S = \#S' + \#S'' = {}_{n-1}C_{k-1} + {}_{n-1}C_k.$$

- Multisets: The number of non-negative solutions $x_1, \ldots, x_n \in \mathbb{N}$ to the equation $x_1 + \cdots + x_n = k$ is
$$\left(\!\!\binom{n}{k}\!\!\right) := \binom{n+k-1}{k}.$$
This is also the number of ways to choose k (unordered) gallons of ice cream from n possible flavors (think of x_i as the number of gallons of flavor i). We could also call these "multisubsets," i.e., subsets with possible repetition.

  Proof: Encode a choice as a binary string containing k copies of 1 and $n-1$ copies of 0:
$$\underbrace{1\cdots1}_{\text{value of } x_1}\ 0\ \underbrace{1\cdots1}_{\text{value of } x_2}\ 0\cdots0\ \underbrace{1\cdots1}_{\text{value of } x_n}$$
The number of such binary strings is $\binom{k+(n-1)}{k}$ because we need to choose k positions to place the 1's from $k+(n-1)$ possible positions. Equivalently, we can choose $n-1$ positions for the 0's. □

- Binomial coefficients are symmetric: $\binom{n}{k} = \binom{n}{n-k}$.

  Counting Proof: Let A be the set of subsets of size k from $\{1, 2, \ldots, n\}$ and let B be the set of subsets of size $n-k$. Then "complementation" is a bijection $A \leftrightarrow B$, hence $\#A = \#B$. Equivalently, let A be the set of binary strings of length n with k copies of 1 and let B be the set of binary strings of length n with $n-k$ copies of 1. Then "flipping all the bits" is a bijection $A \leftrightarrow B$. □

- Substituting $x = 1$ or $x = -1$ into $(1+x)^n = \sum_k \binom{n}{k} x^k$ gives:
$$\begin{aligned} 2^n &= \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}, \\ 0^n &= \binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n\binom{n}{n}, \end{aligned}$$
Differentiating and then substituting $x = 1$ gives:
$$\begin{aligned} n(1+x)^{n-1} &= \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \cdots + n\binom{n}{n}x^{n-1} \\ n2^{n-1} &= \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n}. \end{aligned}$$

- Exercise: Give counting proofs for the three previous identities. For the first identity, group subsets by their number of elements. For the second, flip one bit to obtain a bijection between even and odd subsets. For the third, choose choose a committee and then choose one person from the committee to be the president.

- The Multinomial Theorem says that
$$(a_1 + a_2 + \cdots a_n)^\ell = \sum \binom{\ell}{k_1, k_2, \ldots, k_n} a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n},$$

where the multinomial coefficients are defined by

$$\binom{\ell}{k_1, k_2, \ldots, k_n} = \frac{\ell!}{k_1! k_2! \cdots k_n!}$$

and where the sum is taken over all $k_1, \ldots, k_n \in \mathbb{N}$ such that $k_1 + \cdots k_n = \ell$.

- Substituting $a_1 = \cdots = a_n = 1$ into the multinomial theorem gives

$$n^\ell = \sum \binom{\ell}{k_1, \ldots, k_n}.$$

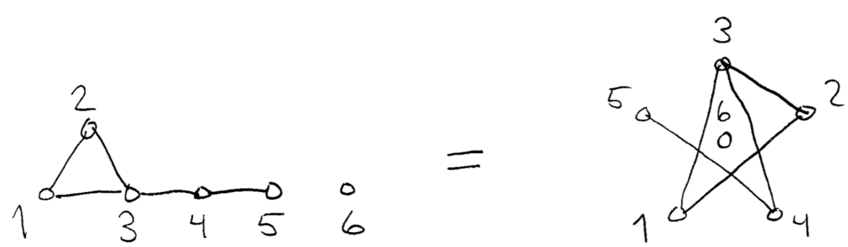
What does this mean? The left side counts the words of length ℓ from the alphabet $\{a_1, \ldots, a_n\}$. The right side counts the same words, but it groups them according to the number of each type of letter. We use the fact that

$$\binom{\ell}{k_1, k_2, \ldots, k_n} = \# \left\{ \begin{array}{c} \text{words of length } \ell \text{ containing} \\ k_i \text{ copies of } a_i \text{ for each } i \end{array} \right\}.$$

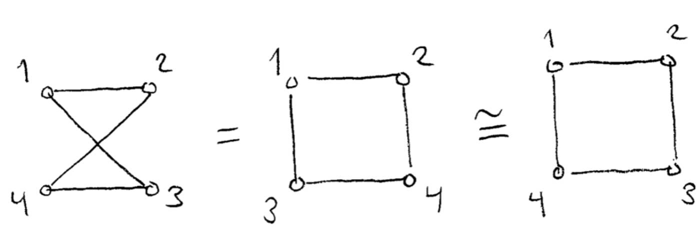
- Example: How many arrangements of the letters $e, f, f, l, o, r, e, s, c, e, n, c, e$?

Topics from Chapter 5

- A simple graph is a set of vertices, together with a set of unordered pairs of vertices, called edges. For example, let $V = \{1, 2, 3, 4, 5, 6\}$ and $E = \{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{3, 4\}, \{4, 5\}\}$.
- It is helpful to draw a graph, but the way you draw it is not important:



- If you permute labels (or if you don't draw labels) then you obtain *isomorphic graphs*:



- To prove that two graphs are isomorphic you must label them. To prove that two graphs are **not** isomorphic you need a trick.
- The easiest trick is to look at the degrees, since these are preserved under isomorphism. Let $G = (V, E)$ be a simple graph. Then for each vertex $u \in V$ we define its degree as

$$\deg(u) := \#\{v \in V : \{u, v\} \in E\}.$$

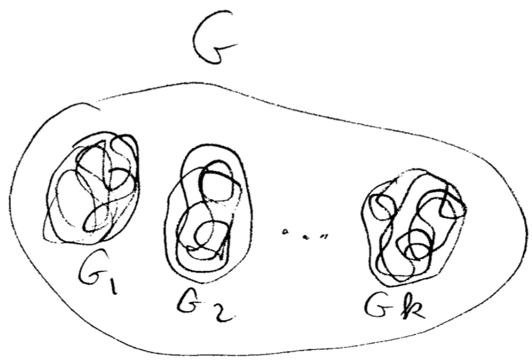
- The Handshaking Lemma says that

$$\sum_{u \in V} \deg(u) = 2 \cdot \#E.$$

  Proof: Let L be the set of lollipops in the graph (a lollipop is an edge together with one of its vertices). By choosing the edge first we have $\#L = 2 \cdot \#E$. By choosing the vertex first we have $\#L = \sum_{u \in V} \deg(u)$. $\square$

- It follows that the number of odd-degree vertices is even. For example, there is no graph with degree sequence $2, 2, 2, 3, 3, 4, 5$ because $2 + 2 + 2 + 3 + 3 + 4 + 5$ is an odd number.
- A graph is called d-regular if each vertex has degree d. If G is a d-regular graph with n vertices then it follows from the First Theorem that dn is even. For example, there does not exist a 3-regular graph on 7 vertices. Exercise: Draw a 3 regular graph on 8 vertices. Exercise: Prove that there exist two non-isomorphic 3-regular graphs on 6 vertices.
- Example: The hypercube Q_n is an n-regular graph on 2^n vertices. The vertices are binary strings of length n and the edges are "bit flips." Exercise: Compute the number of edges.[1]
- Famous graphs include the path P_n, cycle C_n, complete graph K_n and the complete bipartite graph $K_{m,n}$. You should know all the important properties of these graphs and be able to draw them.
- Let $G = (V, E)$ be a simple graph. The complement $\overline{G}$ has the same vertices but the edges and the non-edges have been flipped. Thus if G has n vertices and e edges then $\overline{G}$ has n vertices and $\binom{n}{2} - e$ edges. Exercise: Draw the graph $K_{3,4}$ and its complement.
- A u, v-walk of length ℓ in $G = (V, E)$ is a sequence of vertices $u = v_0, v_1, \ldots, v_\ell = v \in V$ such that $\{v_{i-1}, v_i\} \in E$ for all $i \in \{1, \ldots, \ell\}$. A path is a walk with no repeated vertex. By recursion every u, v-walk contains a u, v-path. Proof: Find a repeated vertex and cut out everything in between. Repeat until there is no repeated vertex.
- We say that the graph is connected if for all $u, v \in V$ there exists a u, v-path. More generally, we define the connected components $G = G_1 \cup G_2 \cup \cdots \cup G_k$ so that vertices $u, v \in V$ are connected if and only if they are in the same component. Picture:

[1] Hao Huang recently (July 1st, 2019) proved the following result, which settled a 30-year-old conjecture: Let A be a subset of vertices in the hypercube Q_n satisfying $\#A \geqslant 2^{n-1} + 1$. Then there exists a vertex $a \in A$ that has at least $\sqrt{n}$ neighbors in A.



- If G has n vertices, e edges and k components then $n-k \leqslant e$. [Remark: This result holds even for multigraphs.]

  Proof by induction on e: Fix $n \geqslant 0$. If $e=0$ then $k=n$ and hence $n-k=0=e$. Now suppose that $e \geqslant 1$ and delete a random edge to obtain a graph G' with n', e', k'. Note that $n'=n$ and $e'=e-1$. Since $e'<e$ we can assume by induction that $n'-k' \leqslant e'$. But we also know that $k' \leqslant k+1$ since deleting an edge creates at most one extra component (and maybe none). Hence

  $$e = e' - 1 \geqslant (n'-k') - 1 = n - 1 - k' \geqslant n - 1 - (k+1) = n-k.$$

  $\square$

- If G is a simple graph with n vertices, e edges and k components then $e \leqslant \binom{n-(k-1)}{2}$. You do not need to prove this. The number of edges is maximized when every component but one is a single vertex and the last component is a complete graph on $n-(k-1)$ vertices.

- A circuit is a walk that begins and ends at the same vertex. A cycle is a circuit that has no repeated vertices (except for the basepoint). Every circuit contains a cycle.

- First Application: A graph is called bipartite if it has **no odd cycles**. Equivalently, we can color the vertices with two colors such that no two vertices of the same color share an edge. (You don't need to know the proof.)

- Second Application: A graph is called a forest if it has **no cycles at all**. One can show that this happens exactly when $e=n-k$, i.e., when the number of edges is minimized. A forest with one connected component ($k=1$) is called a tree. In other words, a tree is a connected graph with no cycles. Equivalently, a tree is a connected graph on n vertices with $e=n-1$ edges. Exercise: Draw a forest with $n=12$ and $k=3$. Verify that the number of edges is $e=n-k=9$.

- Let G be a tree on vertex set $\{1,2,\ldots,n\}$ and let $d_i := \deg(i)$. Since G has $e=n-1$ edges we must have

  $$\sum_{i=1}^{n} d_i = 2(n-1)$$

and hence

$$\sum_{i=1}^{n}(d_i - 1) = \sum_{i=1}^{n} d_i - \sum_{i=1}^{n} 1 = 2(n-1) - n = 2n - 2 - n = n - 2.$$

- Cayley's Tree Formula says that

$$\binom{n-2}{d_1 - 1, d_2 - 1, \ldots, d_n - 1} = \# \left\{ \begin{array}{c} \text{trees on vertex set } \{1, \ldots, n\} \\ \text{where vertex } i \text{ has degree } d_i \end{array} \right\}.$$

  By summing over all possible degrees we obtain

$$\#\{\text{labeled trees on } n \text{ vertices}\} = \sum \binom{n-2}{d_1 - 1, d_2 - 1, \ldots, d_n - 1} = n^{n-2}.$$

  Exercise: Verify that this last step follows from the multinomial theorem.

- Prüfer's proof of Cayley's Formula: Given a tree T on $\{1, 2, \ldots, n\}$, delete the smallest leaf (vertex of degree one) and let p_1 be the name of its parent. Repeat to obtain a sequence $(p_1, p_2, \ldots, p_{n-2})$ called the *Prüfer code* of the tree. One can show that every word of length $n-2$ from the alphabet $\{1, \ldots, n\}$ is the Prüfer code of some tree. (You don't need to show this.) Furthermore, the number i shows up exactly $d_i - 1$ times in the code. Example: