Math 309 **Summer B 2018**
Homework 3 **Drew Armstrong**

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1. Poker Hands. A standard deck contains 52 cards. Half the cards are red and half are black. A "poker hand" consists of 5 cards chosen at random from the deck.

(a) How many different poker hands are there?
(b) How many poker hands contain all red cards?
(c) How many poker hands contain 1 red and 4 black cards? [Hint: Choose the red card first, then choose the black cards.]

2. Double Counting. In this problem you will give two proofs of the identity

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

(a) Prove the identity using pure algebra. [Hint: $n! = n \times (n-1)!$]
(b) In a certain classroom of n students we want to choose a committee of k students, one of which will be the president of the committee. Prove the identity by counting the possible choices in two different ways. [Hint: Will you choose the president before or after choosing the committee members?]

3. Trinomial Coefficients. Consider integers $i, j, k \geq 0$ such that $i + j + k = n$, and let N be the number of words that can be made with the letters

$$\underbrace{a, a, \ldots, a}_{i \text{ copies}}, \underbrace{b, b, \ldots, b}_{j \text{ copies}}, \underbrace{c, c, \ldots, c}_{k \text{ copies}}.$$

(a) Explain why $n! = N \times i! \times j! \times k!$.
(b) How many words can be made from the letters b, a, n, a, n, a?

4. Falling Factorial. For any number z and for any integer $k \geq 0$ we define the "falling factorial" notation $(z)_k := z(z-1)(z-2)\cdots(z-k+1)$. If $n \geq 0$ is an integer, show that

$$\binom{n}{k} = \frac{(n)_k}{k!}.$$

5. Newton's Binomial Theorem. Consider any integer $k \geq 0$. Based on Problem 4, Isaac Newton defined the notation

$$\binom{z}{k} := \frac{(z)_k}{k!}$$

for **any number** z (not just positive whole numbers), and he showed that for any number x with $|x| < 1$ the following infinite series is convergent:

$$(1+x)^z = 1 + \binom{z}{1}x + \binom{z}{2}x^2 + \binom{z}{3}x^3 + \cdots.$$

(a) For any integers $n, k \geq 1$ show that

$$\binom{-n}{k} = (-1)^k\binom{n+k-1}{k}.$$

(b) Use Newton's formula to obtain an infinite series expansion for $(1+x)^{-2}$.