Math 309 **Summer B 2018**
Homework 1 **Drew Armstrong**

For any positive integers p and n, we let $S_p(n)$ denote the sum of the first n "p-th powers:"

$$S_p(n) := \sum_{i=1}^{n} i^p = 1^p + 2^p + 3^p + \cdots + n^p.$$

1. In class I gave a proof that

$$(*) \qquad S_1(n) = \frac{n(n+1)}{2}.$$

Now you will give a different proof.

(a) Show that equation $(*)$ is true when $n = 1$.
(b) Let k be an arbitrary positive integer and **assume** that equation $(*)$ is true when $n = k$. In this case show that $(*)$ must also be true when $n = k + 1$. [Hint: Use the fact that $S_1(k + 1) = S_1(k) + (k + 1)$.]

(a) **Base Case:** When $n = 1$ we observe that $1 = 1(2)/2$ is a true statement.
(b) **Induction Step:** Consider any integer $k \geq 1$ and assume for induction that the statement $(*)$ is true when $n = k$. In other words, assume that $S_1(k) = k(k + 1)/2$ is a true statement. In this (hypothetical) case, we must have

$$\begin{aligned} S_1(k+1) &= 1 + 2 + \cdots + (k+1) \\ &= (1 + 2 + \cdots + k) + (k+1) \\ &= S_1(k) + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) && \text{(assumption)} \\ &= (k+1)\left[\frac{k}{2} + 1\right] \\ &= (k+1)\frac{(k+2)}{2} \\ &= \frac{(k+1)[(k+1)+1]}{2}. \end{aligned}$$

Hence the statement $(*)$ is also true when $n = k + 1$. □

2. In class I gave a proof that

$$(**) \qquad S_2(n) = \frac{n(n+1)(2n+1)}{6}.$$

Now you will give a different proof.

(a) Show that equation $(**)$ is true when $n = 1$.
(b) Let k be an arbitrary positive integer and **assume** that equation $(**)$ is true when $n = k$. In this case show that $(**)$ must also be true when $n = k + 1$. [Hint: Use the fact that $S_2(k + 1) = S_2(k) + (k + 1)^2$.]

(a) **Base Case:** When $n = 1$ we observe that $1^2 = 1(2)(3)/6$ is a true statement.
(b) **Induction Step:** Consider any integer $k \geq 1$ and assume for induction that the statement $(**)$ is true when $n = k$. In other words, assume that $S_2(k) = k(k+1)(2k+1)/2$ is a true statement. In this (hypothetical) case, we must have

$$\begin{aligned}
S_2(k+1) &= 1^2 + 2^2 + \cdots + (k+1)^2 \\
&= (1^2 + 2^2 + \cdots + k^2) + (k+1)^2 \\
&= S_2(k) + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 && \text{(assumption)} \\
&= (k+1)\left[\frac{k(2k+1)}{6} + (k+1)\right] \\
&= (k+1)\left[\frac{2k^2+k}{6} + \frac{6k+6}{6}\right] \\
&= (k+1)\frac{(2k^2+7k+6)}{6} \\
&= (k+1)\frac{(k+2)(2k+3)}{6} \\
&= \frac{(k+1)\cdot[(k+1)+1]\cdot[2(k+1)+1]}{6}.
\end{aligned}$$

Hence the statement $(**)$ is also true when $n = k+1$. $\square$

3. (Steiner's Problem) Suppose that we have a round pizza and let L_n be the maximum number of pieces we can obtain from n straight cuts. We proved in class that

$$L_n = 1 + (1 + 2 + 3 + \cdots + n) = 1 + \frac{n(n+1)}{2} = \frac{n^2+n+2}{2}.$$

Now suppose we have a round ball of cheese and let P_n be the maximum number of pieces we can obtain from n flat cuts. You may assume without proof that we have

$$\boxed{P_{n+1} = P_n + L_n \quad \text{for all } n \geq 0.}$$

(a) Use this recurrence to show that for all $n \geq 0$ we have

$$P_{n+1} = 1 + L_0 + L_1 + L_2 + \cdots L_n = 1 + \sum_{k=0}^{n} L_k = 1 + \sum_{k=0}^{n}\left(\frac{k^2+k+2}{2}\right).$$

(b) Simplify the expression in part (a) to show that

$$P_{n+1} = \frac{(n+2)(n^2+n+6)}{6},$$

and hence

$$P_n = \frac{(n+1)(n^2-n+6)}{6}.$$

[Hint: Use the results from Problems 1 and 2.]

There's not much to do for part (a). For part (b) we have

$$\begin{aligned}
P_{n+1} &= 1 + \sum_{k=0}^{n} \left(\frac{k^2+k+2}{2} \right) \\
&= 1 + \frac{1}{2}\left(\sum_{k=0}^{n} k^2 \right) + \frac{1}{2}\left(\sum_{k=0}^{n} k \right) + \left(\sum_{k=0}^{n} 1 \right) \\
&= 1 + \frac{1}{2}\left(\sum_{k=1}^{n} k^2 \right) + \frac{1}{2}\left(\sum_{k=1}^{n} k \right) + \left(\sum_{k=0}^{n} 1 \right) \\
&= 1 + \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6} \right) + \frac{1}{2}\left(\frac{n(n+1)}{2} \right) + (n+1) \\
&= \frac{12}{12} + \frac{2n^3+3n^2+n}{12} + \frac{3n^3+3n}{12} + \frac{12n+12}{12} \\
&= \frac{2n^3+6n^2+16n+24}{12} \\
&= \frac{n^3+3n^2+8n+12}{6} \\
&= \frac{(n+2)(n^2+n+6)}{6}.
\end{aligned}$$

[Remark: I don't really care about the factorization in the last step. My computer did it for me.] Then by substituting $n \to (n-1)$ we conclude that

$$P_n = \frac{[(n-1)+2]\cdot[(n-1)^2+(n-1)+6]}{6} = \frac{(n+1)(n^2-n+6)}{6}.$$

For example, if we cut a round cheese 5 times, the maximum number of pieces we can get is

$$P_5 = \frac{6\cdot(25-5+6)}{6} = 26.$$

That would be very hard to figure out with pictures.

[Remark: Jakob Steiner solved this problem in 1826. Later in the 1840s, Ludwig Schläfli suggested to rewrite Steiner's formula in terms of binomial coefficients:[1]

$$P_n = \frac{n(n-1)(n-2)}{6} + \frac{n(n-1)}{2} + n + 1 = \binom{n}{3} + \binom{n}{2} + \binom{n}{1} + \binom{n}{0}.$$

At the same time, Schläfli showed that the maximum number of pieces of a "d-dimensional hypercheese" that can be obtained from n flat cuts is

$$\binom{n}{d} + \binom{n}{d-1} + \cdots + \binom{n}{2} + \binom{n}{1} + \binom{n}{0},$$

whatever that means.]

[1] We will talk about these later.