Math 309 **Summer B 2018**
Homework 1 **Drew Armstrong**

For any positive integers p and n, we let $S_p(n)$ denote the sum of the first n "p-th powers:"

$$S_p(n) := \sum_{i=1}^{n} i^p = 1^p + 2^p + 3^p + \cdots + n^p.$$

1. In class I gave a proof that

$$(*) \qquad S_1(n) = \frac{n(n+1)}{2}.$$

Now you will give a different proof.

(a) Show that equation $(*)$ is true when $n = 1$.

(b) Let k be an arbitrary positive integer and **assume** that equation $(*)$ is true when $n = k$. In this case show that $(*)$ must also be true when $n = k+1$. [Hint: Use the fact that $S_1(k+1) = S_1(k) + (k+1)$.]

2. In class I gave a proof that

$$(**) \qquad S_2(n) = \frac{n(n+1)(2n+1)}{6}.$$

Now you will give a different proof.

(a) Show that equation $(**)$ is true when $n = 1$.

(b) Let k be an arbitrary positive integer and **assume** that equation $(**)$ is true when $n = k$. In this case show that $(**)$ must also be true when $n = k+1$. [Hint: Use the fact that $S_2(k+1) = S_2(k) + (k+1)^2$.]

3. (Steiner's Problem) Suppose that we have a round pizza and let L_n be the maximum number of pieces we can obtain from n straight cuts. We proved in class that

$$L_n = 1 + (1+2+3+\cdots+n) = 1 + \frac{n(n+1)}{2} = \frac{n^2+n+2}{2}.$$

Now suppose we have a round ball of cheese and let P_n be the maximum number of pieces we can obtain from n flat cuts. You may assume without proof that we have

$$\boxed{P_{n+1} = P_n + L_n \quad \text{for all } n \geq 0.}$$

(a) Use this recurrence to show that for all $n \geq 0$ we have

$$P_{n+1} = 1 + L_0 + L_1 + L_2 + \cdots L_n = 1 + \sum_{k=0}^{n} L_k = 1 + \sum_{k=0}^{n} \left(\frac{k^2+k+2}{2} \right).$$

(b) Simplify the expression in part (a) to show that

$$P_{n+1} = \frac{(n+2)(n^2+n+6)}{6},$$

and hence

$$P_n = \frac{(n+1)(n^2-n+6)}{6}.$$

[Hint: Use the results from Problems 1 and 2.]