

6/30/14

Welcome to MTH 309

Course Info:

Drew Armstrong

armstrong@math.miami.edu

office: Ungar 583

There is no required textbook.

I will scan all course notes and post them on my webpage:

[www.math.miami.edu/~armstrong](http://www.math.miami.edu/~armstrong)

Your grade is based on

$\frac{1}{3}$  Weekly Homework (Fridays)

$\frac{1}{3}$  Weekly Quiz (Mondays)

$\frac{1}{3}$  Final Exam

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1

Course Topic: Discrete Mathematics

"Discrete" is the opposite of "Continuous"  
(Calculus)

This is particularly relevant for computer applications: any continuous problem must first be "discretized" before it can be implemented on a computer.

Some major themes:

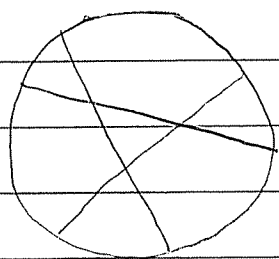
- Recursion / Induction
- Counting and Probability
- Number Theory and Cryptography
- What Do You Want to See?

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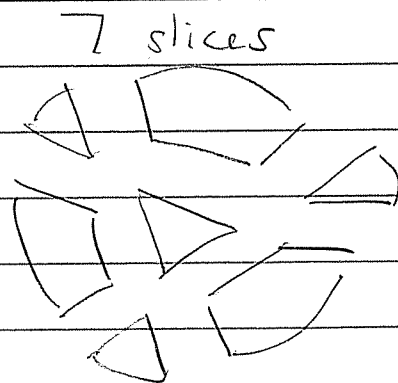
Before we begin, let's look at a sample problem.

Problem (Jacob Steiner, 1826):

What is the maximum number of slices of pizza you can get by making  $n$  straight cuts?



~~~~~  
3 cuts



More formally, let

$L_n$  := The maximum number of regions cut out by  $n$  (infinite) lines in the (infinite) plane.

[ Remark: The symbol " $:=$ " means "is defined to be". ]

Our Goal: Solve for  $L_n$ .

What does that mean?

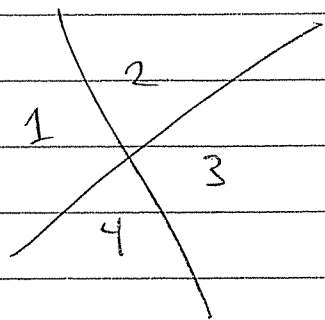
- Say how to compute  $L_n$ .
- Say how to compute  $L_n$  efficiently.
- Give a "formula" for  $L_n$ .
- Give a "nice formula" for  $L_n$ .
- ⋮

we'll see

To analyze a discrete problem we always start with experiments.

Obviously  $L_0 = 1$

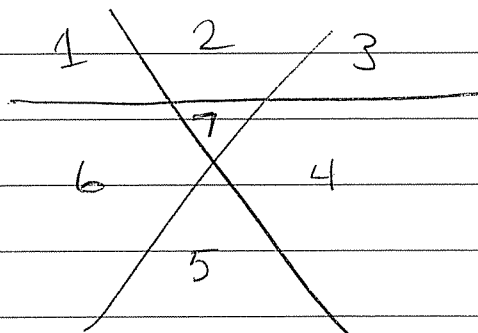
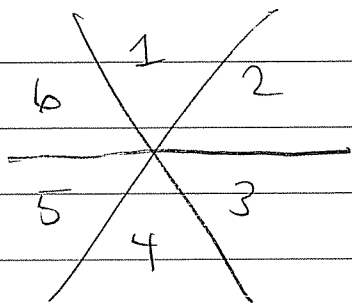
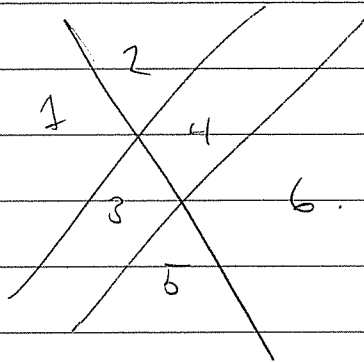
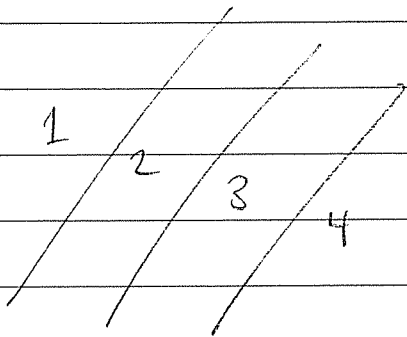
$L_1 = 2$



$L_2 = 4$

[ Guess ("Conjecture") :  $L_n = 2^n$  ? ]

$L_3 = ?$



It looks like  $L_3 = 7$

So our guess was wrong. ☹️

$$L_3 = 7 \neq 2^3 = 8$$

Now you do some experiments

(wait)

Table of observations

|       |   |   |   |   |    |    |    |     |
|-------|---|---|---|---|----|----|----|-----|
| n     | 0 | 1 | 2 | 3 | 4  | 5  | 6  | ... |
| $L_n$ | 1 | 2 | 4 | 7 | 11 | 16 | 22 | ... |

Can we come up with a theory to explain these observations?

(wait)

Here's the key observation:

$$L_n = L_{n-1} + n$$

for  $n > 0$

There are two issues:

① Why is this true?

② Can we use it to solve the problem?

Let's do ② first because it's more interesting.

We have a recurrence relation with an initial condition

- $L_0 = 1$

- $L_n = L_{n-1} + n$  for  $n > 0$ .

We can expand this out:

$$L_1 = L_0 + 1 = 1 + 1$$

$$L_2 = L_1 + 2 = 1 + 1 + 2$$

$$L_3 = L_2 + 3 = 1 + 1 + 2 + 3$$

$$L_4 = L_3 + 4 = 1 + 1 + 2 + 3 + 4$$

$$L_n = 1 + (1 + 2 + \dots + n)$$

$$L_n = 1 + \sum_{k=1}^n k \leftarrow \text{the sum of } k \text{ as } k \text{ goes from } 1 \text{ to } n$$

Yay. We have a "formula" for  $L_n$ :

$$L_n = 1 + \sum_{k=1}^n k.$$

But is it a "good formula"?

(Maybe we can't do any better?)

This problem is nice because there's a lucky trick.

$$\text{Let } S_n := \sum_{k=1}^n k = 1 + 2 + 3 + \dots + n.$$

Theorem (Carl Friedrich Gauss, when he was 6 years old):

$$S_n = \frac{n(n+1)}{2}$$

Proof: Consider the quantity  $2 \cdot S_n$ .

$$2S_n = S_n + S_n$$

$$= (1 + 2 + 3 + \dots + n) + (n + (n-1) + (n-2) + \dots + 1).$$

$$= (n+1) + (n+1) + (n+1) + \dots + (n+1).$$

n times

$$\text{So } 2S_n = n(n+1)$$

$$\Rightarrow S_n = \frac{n(n+1)}{2}$$

This gives us a "closed formula" for  $L_n$ .

$$\begin{aligned} L_n &= 1 + S_n \\ &= 1 + \frac{n(n+1)}{2} \\ &= \frac{2 + n(n+1)}{2} \\ &= \frac{n^2 + n + 2}{2} \end{aligned}$$

Conclusion:

$$L_n = \frac{n^2 + n + 2}{2}$$

Is that a "nice" formula? Yes!

[Thinking Homework: Why is  $n^2 + n + 2$  always an even number?]



Bonus: We get an asymptotic estimate.  
For large  $n$  we have

$$L_n \sim \frac{1}{2} n^2$$

$$\left[ \text{Meaning: } \lim_{n \rightarrow \infty} \frac{L_n}{\frac{1}{2} n^2} = 1. \right]$$

That might be useful.

~~WE RAN OUT OF TIME HERE.~~

Now we return to (1)

Why is  $L_n = L_{n-1} + n$ ? We want a "proof".

Imagine we have successfully divided the plane into  $L_n$  regions using  $n$  lines

$$l_1, l_2, l_3, \dots, l_n.$$

Now remove line  $l_n$ . How many regions do we have just using the lines

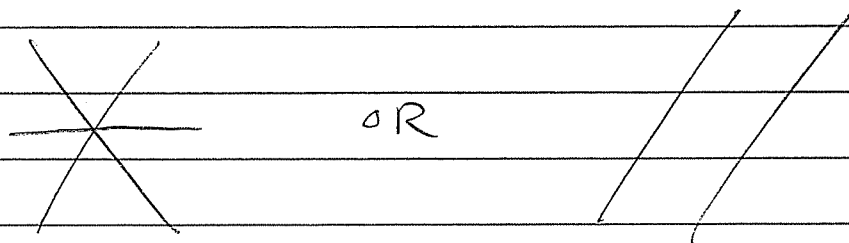
$$l_1, l_2, \dots, l_{n-1} \quad ?$$

Answer:  $L_{n-1}$  regions.

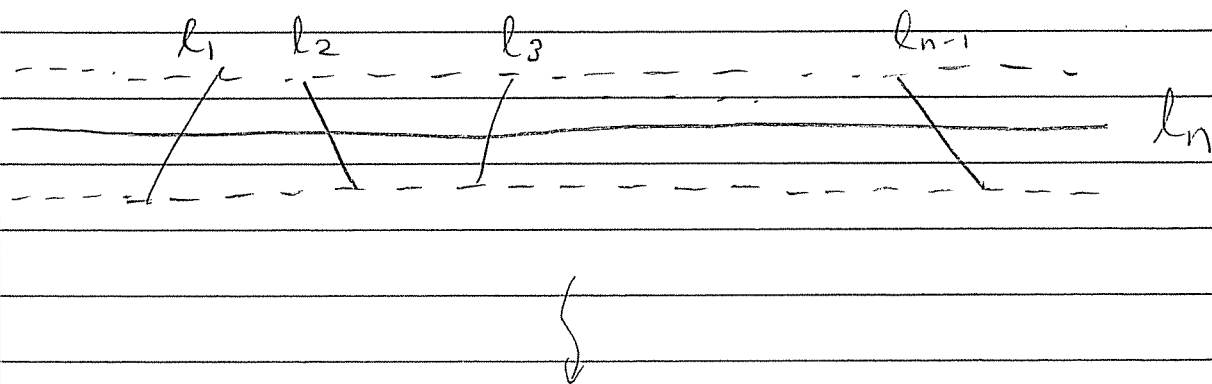
[Reason: We must get the maximum possible number of regions because  $L_n$  is by definition maximal. Don't worry too much about this.]

Okay, now what happens to these  $L_{n-1}$  regions when we add line  $l_n$  back in?

Observe that we never have

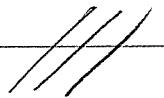


(otherwise we don't get the maximum number of regions). So a small neighborhood of the line  $l_n$  looks like

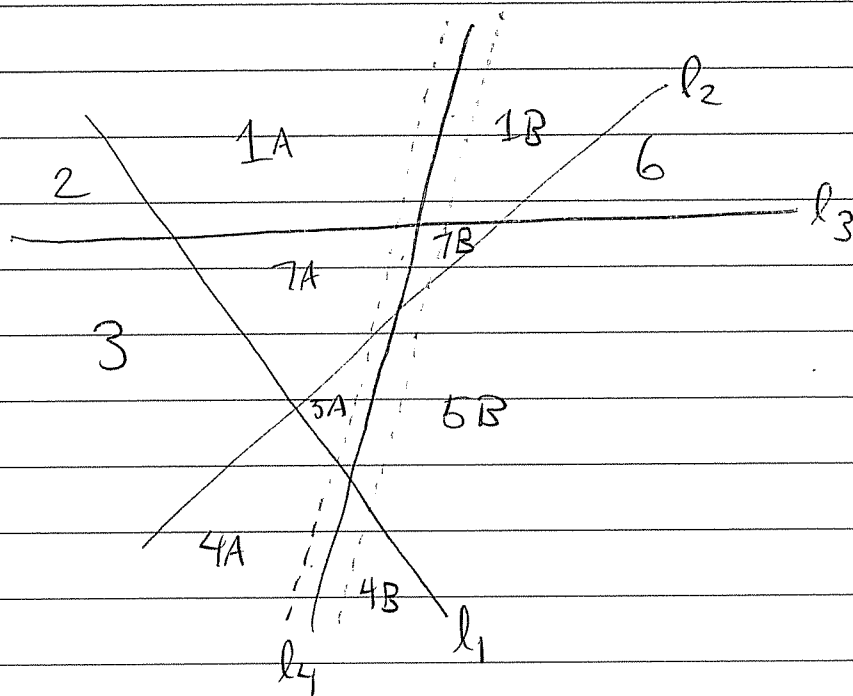


That is, line  $l_n$  passes through exactly  $n$  of the original  $L_{n-1}$  regions and divides each of these  $n$  regions into 2. Thus we get  $n$  new regions:

$$L_n = L_{n-1} + n$$



If you don't believe it, here's a picture:



Line  $l_4$  creates 4 new regions.

## Problems for Thinking:

1. Explain why

$$\lim_{n \rightarrow \infty} \frac{\ln n}{\frac{1}{2} n^2} = 1$$

2. Is there a "closed formula" for the sum of the first  $n$  squares?

$$\sum_{k=1}^n k^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 = ?$$

3. Jacob Steiner (1826) also considered the problem of dividing 3-dim space by  $n$  2-dim planes

Let  $P_n$  be the maximum number of 3-dim regions you can get.

- Do you think  $P_n$  has a closed formula?
- How would you approach the problem?
- Roughly how big is  $P_n$ ?

7/1/14

- HW 1 due at the beginning of Friday's class
- Quiz 1 at the beginning of Monday's class.

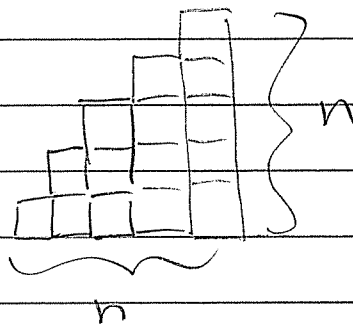
Last time we saw "Gauss' Trick".

$$\text{Let } S_n := \sum_{k=1}^n k = 1 + 2 + 3 + \dots + n$$

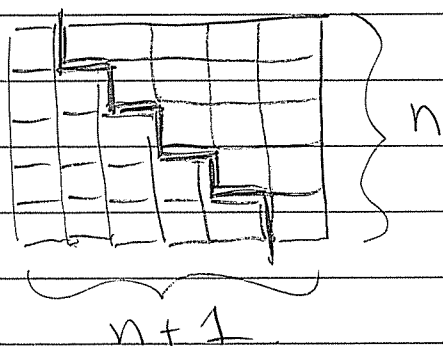
$$\text{Then } S_n = \frac{n(n+1)}{2}$$

Proof: observe that  $S_n$  is the area of a staircase made of  $1 \times 1$  squares.

$S_n = \text{area of}$



Thus  $2S_n$  is the area of two staircases put together.



This is a rectangle of area  $n(n+1)$ .

Hence  $2S_n = n(n+1)$ .

$$\Rightarrow S_n = \frac{n(n+1)}{2}$$

Now let's consider the sum of the first  $n$  squares.

$$\text{Let } \square_n := \sum_{k=1}^n k^2 = 1^2 + 2^2 + 3^2 + \dots + n^2.$$

Q: Does this have a nice formula?

A: Yes, but it's harder to find.

Theorem: For all  $n \geq 0$  we have

$$\square_n = \frac{n(n+1)(2n+1)}{6}$$

Proof (It's a trick; don't worry if you wouldn't think of this)

We will temporarily consider the sum of the first  $n$  cubes.

$$\text{Let } \text{Cube}_n := \sum_{k=1}^n k^3 = 1^3 + 2^3 + 3^3 + \dots + n^3$$

Now we will look at the sum  $\sum_{k=1}^{n+1} k^3$  in two different ways.

On one hand we have

$$\begin{aligned}\sum_{k=1}^{n+1} k^3 &= \sum_{k=1}^{n+1} k^3 = \underbrace{1^3 + 2^3 + \dots + n^3}_{\sum_{k=1}^n k^3} + (n+1)^3 \\ &= \sum_{k=1}^n k^3 + (n+1)^3 \\ &= \sum_{k=1}^n k^3 + (n+1)^3\end{aligned}$$

On the other hand we have

$$\begin{aligned}\sum_{k=1}^{n+1} k^3 &= \sum_{k=1}^{n+1} k^3 = \sum_{k=0}^n (k+1)^3 \\ &= \sum_{k=0}^n (k^3 + 3k^2 + 3k + 1) \\ &= \sum_{k=0}^n k^3 + 3 \cdot \sum_{k=0}^n k^2 + 3 \sum_{k=0}^n k + \sum_{k=0}^n 1 \\ &= \sum_{k=1}^n k^3 + 3 \cdot \sum_{k=1}^n k^2 + 3 \cdot \sum_{k=1}^n k + \sum_{k=0}^n 1 \\ &= \sum_{k=1}^n k^3 + 3 \cdot \sum_{k=1}^n k^2 + 3 \cdot S_n + (n+1)\end{aligned}$$

Equating both expressions for  $\square_{n+1}$  gives

$$\cancel{\square_n} + (n+1)^3 = \cancel{\square_n} + 3 \cdot \square_n + 3 \cdot S_n + (n+1).$$

Note that  $\square_n$  cancels. (That's the trick!)

Now we can solve for  $\square_n$ :

$$(n+1)^3 = 3 \square_n + 3 S_n + (n+1)$$

$$3 \square_n = (n+1)^3 - 3 S_n - (n+1)$$

$$= (n+1)^3 - 3 \frac{n(n+1)}{2} - (n+1)$$

$$= (n+1) \left( \frac{(n+1)^2 - 3n - 1}{2} \right)$$

$$= (n+1) \left( \frac{2(n+1)^2 - 3n - 2}{2} \right)$$

$$= \frac{1}{2} (n+1) (2n^2 + 4n + 2 - 3n - 2)$$

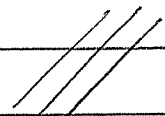
$$= \frac{1}{2} (n+1) (2n^2 + n)$$

$$= \frac{1}{2} n(n+1)(2n+1).$$



Hence

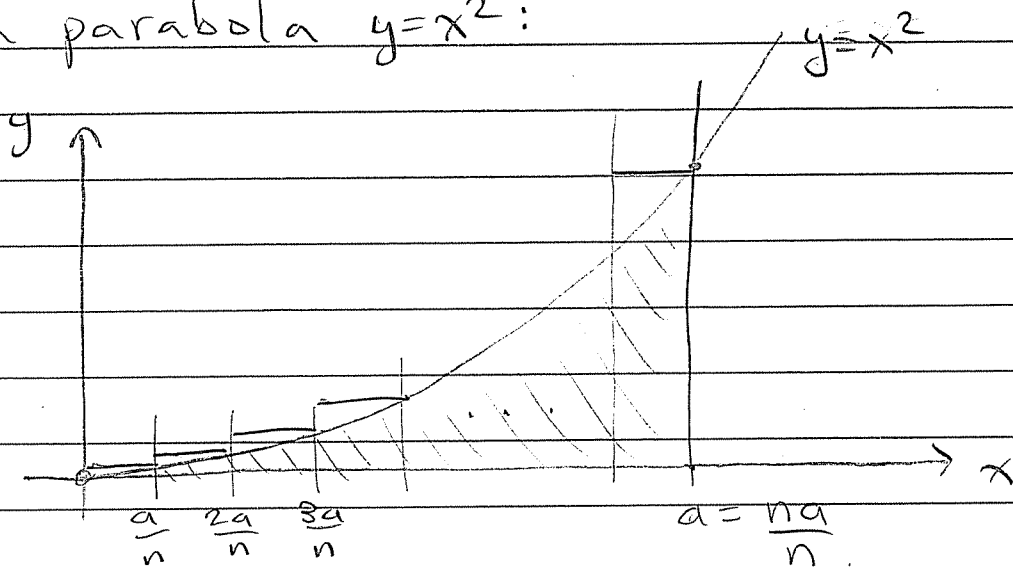
$$\square_n = \frac{1}{6} n(n+1)(2n+1)$$



Q: Why would anyone care?

A: Here's why Pierre de Fermat cared (1637).

He wanted to compute the area under a parabola  $y=x^2$ :



$$\text{Exact area} = \int_0^a x^2 dx$$

$\approx$  sum of areas of rectangles

$$= \sum_{k=1}^n \underbrace{\frac{a}{n}}_{\text{base}} \underbrace{\left(\frac{ka}{n}\right)^2}_{\text{height}}$$

$$= \sum_{k=1}^n \frac{a}{n} \frac{k^2 a^2}{n^2}$$

$$= \frac{a^3}{n^3} \cdot \sum_{k=1}^n k^2$$

$$= \frac{a^3}{n^3} \cdot \frac{1}{6} n(n+1)(2n+1)$$

$$= \frac{a^3}{n^3} \left( \frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{1}{6} n \right)$$

$$= a^3 \left( \frac{1}{3} + \left[ \frac{1}{2n} + \frac{1}{6n^2} \right] \right)$$

↑  
this goes to 0 as  $n \rightarrow \infty$ .

Taking the limit as  $n \rightarrow \infty$  gives

$$\int_0^a x^2 dx = \lim_{n \rightarrow \infty} a^3 \left( \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} \right)$$

$$= a^3 \left( \frac{1}{3} + 0 + 0 \right)$$

$$= \frac{a^3}{3} \quad \text{☺}$$

Of course the Fundamental Theorem of Calculus would be faster, but it was not known in 1637.

Remarks:

1. Actually, Fermat only needed the asymptotic estimate

$$\square_n \sim \frac{1}{3} n^3$$

2. Consider the sum of  $p$ th powers:

$$f_p(n) := \sum_{k=1}^n k^p$$

Fermat proved that  $f_p(n) \sim \frac{1}{p+1} n^{p+1}$

and hence  $\int_0^a x^p dx = \frac{1}{p+1} a^{p+1}$ .

3. We know  $\square_n \sim \frac{1}{4} n^4$ .

Can you give an exact formula?

BREAK / WE RAN OUT OF TIME AGAIN!

Recall from last time:

$L_n$  := The maximum number of regions formed by  $n$  lines in the plane.

We guessed that

$$L_n = L_{n-1} + n \quad \text{for all } n > 0$$

but we never "proved" it. In other words,

Q: Why is it true that

$$L_n = L_{n-1} + n \quad ?$$

Here's an explanation ("proof"):

Imagine we have successfully divided the plane into  $L_n$  regions using the  $n$  lines

$$l_1, l_2, l_3, \dots, l_n.$$

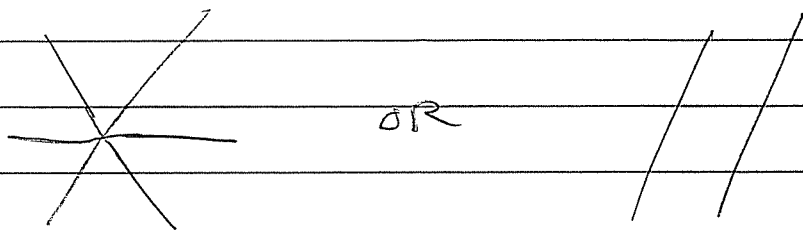
Now remove the line  $l_n$ . How many regions are formed by the lines

$l_1, l_2, \dots, l_{n-1}$  ?

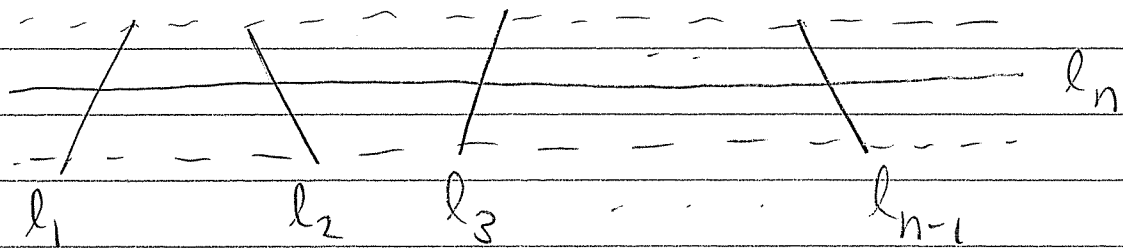
Answer: It must be the maximum, i.e.,  $L_{n-1}$ . It couldn't be any fewer because we started with the maximum  $L_n$ . (This is the most subtle part of the proof; don't worry too much.)

Now we add the line  $l_n$  back in to get some information about  $L_n$ .

Observe that we never have



because we assumed we have a maximum number of regions. Hence a small neighborhood of the line  $l_n$  must look like this:



This means that  $l_n$  passes through exactly  $n$  of the regions defined by  $l_1, \dots, l_{n-1}$ , and cuts each of them in 2.

Hence the number of regions defined by  $l_1, l_2, \dots, l_n$  is

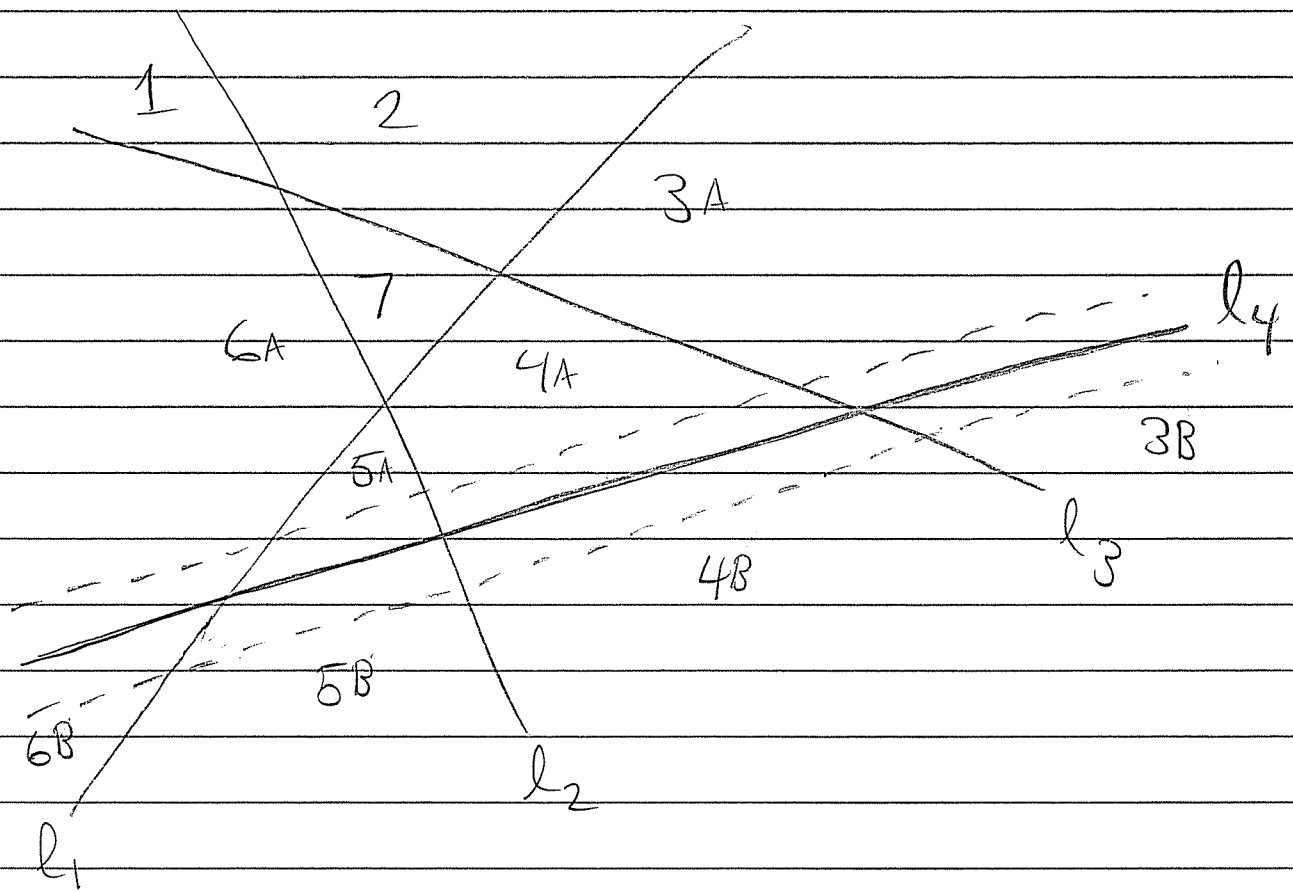
$$L_{n-1} + n.$$

On the other hand, we began by assuming that the number of regions is  $L_n$ .

Hence  $L_n = L_{n-1} + n.$

[Remark: There is more going on here than meets the eye. It takes a lot of work to appreciate the subtleties involved.]

While it is not strictly part of the proof, an example is necessary for our intuition.



Adding the line  $l_4$  creates exactly 4 new regions (the regions 3, 4, 5, 6 become

3A, 4A, 5A, 6A, 3B, 4B, 5B, 6B.)

Question for Thinking:

How many finite (ie. bounded) regions are formed by  $n$  lines in the plane?

Can you prove it?

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Having failed to prove

$$L_n = L_{n-1} + n$$

twice, we will discuss it thoroughly tomorrow (I hope.)