# Math 309 — Summer 2014
# Homework 2 Solutions — Drew Armstrong

If $S$ is a **finite set** then we let $\#S$ denote its number of elements. We call this the **size** or the **cardinality** of $S$. Sometimes we will use the equivalent notation $|S| := \#S$.

**1.** If $S$ and $T$ are finite sets, what is the size of the Cartesian product $S \times T$?

*Proof.* I claim that the Cartesian product has size $\#(S \times T) = \#S \times \#T$. To see this, we will **name** the elements of the sets as follows:

$$S := \{s_1, s_2, \ldots, s_m\} \qquad T := \{t_1, t_2, \ldots, t_n\}.$$

Observe that this notation implies $m = \#S$ and $n = \#T$. Now observe that an element of the Cartesian product is just an element of the following "rectangle" whose rows are indexed by the elements of $S$ and whose columns are inexed by the elements of $T$:

| | $t_1$ | $t_2$ | $\cdots$ | $t_n$ |
|---|---|---|---|---|
| $s_1$ | $(s_1, t_1)$ | $(s_1, t_2)$ | $\cdots$ | $(s_1, t_n)$ |
| $s_2$ | $(s_2, t_1)$ | $(s_2, t_2)$ | $\cdots$ | $(s_2, t_n)$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| $s_m$ | $(s_m, t_1)$ | $(s_m, t_2)$ | $\cdots$ | $(s_m, t_n)$ |

And how many elements does this rectangle have? Isn't this just the **definition** of $m \times n$? (Yes it is.) We conclude that

$$\#(S \times T) = m \times n = \#S \times \#T.$$

$\square$

**2.** If $S$ and $T$ are finite sets, how many different functions are there from $S$ to $T$? Express your answer in terms of the numbers $\#S$ and $\#T$.

*Proof.* Recall that a function from $S$ to $T$ is a set of arrows from $S$ to $T$ (in other words, a subset of $S \times T$) satisfying **one** axiom:

- Each element of $S$ has exactly one arrow pointing from it.

So if $S$ is finite then a function from $S$ to $T$ consists of exactly $\#S$ arrows. To specify the function we need to say where each of these arrows points. If $T$ is finite, then each of the $\#S$ arrows has exactly $\#T$ possibilities for where it points. These choices can be made completely independently, and so the total number of possibilities is

$$\underbrace{\#T \times \#T \times \cdots \times \#T}_{\#S \text{ times}} = \#T^{\#S}.$$

We conclude that the number of different functions from $S$ to $T$ is $\#T^{\#S}$. For this reason we might sometimes use the **cute** notation $T^S$ for the **set** of different functions from $S$ to $T$. Do you like this notation? $\square$

**3.** Apply your answers from Problems 1 and 2 to show that there are 16 possible functions from the set $\{T, F\}^2 := \{T, F\} \times \{T, F\}$ to the set $\{T, F\}$.

*Proof.* To count the functions from $\{T, F\}^2$ to $\{T, F\}$ we let $S := \{T, F\}^2$ and $T := \{T, F\}$. Note that $\#T = 2$, and by Problem 1 we have

$$\#S = \#\{T, F\}^2 = \#\{T, F\} \times \#\{T, F\} = 2 \times 2 = 4.$$

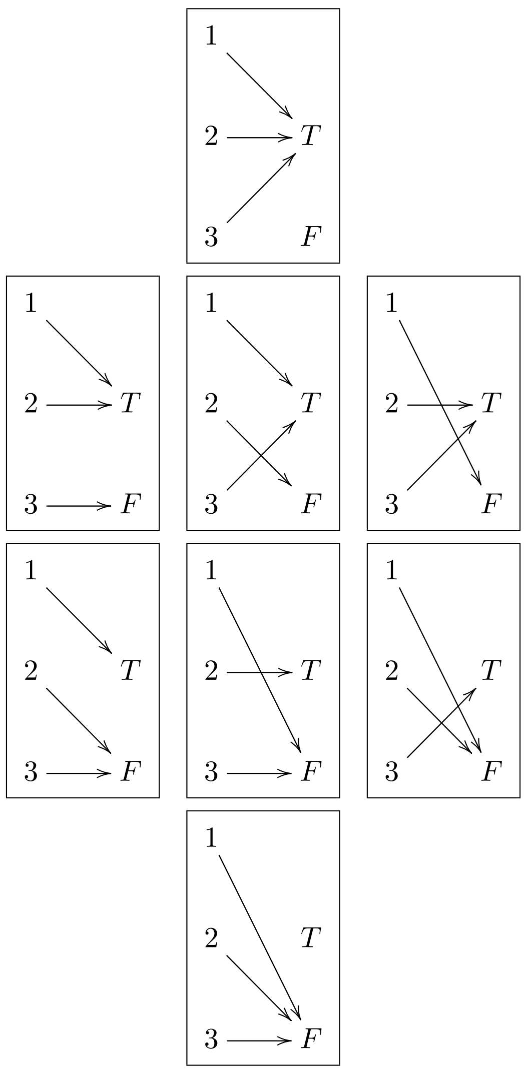
Then by applying Problem 2, we see that the total number of functions $S \to T$ is

$$\#T^{\#S} = 2^4 = 16.$$

$\square$

**4.** Explicitly write down all of the functions from $\{1, 2, 3\}$ to $\{T, F\}$.

*Proof.* Here they are. There are $2^3 = 8$ of them, as expected.




$\square$

**5.** Explicitly write down all of the subsets of $\{1,2,3\}$.

*Proof.* Here they are.

$$\{1,2,3\}$$

$$\{1,2\} \quad \{1,3\} \quad \{2,3\}$$

$$\{1\} \quad \{2\} \quad \{3\}$$

$$\emptyset$$

Can anyone see why I arranged them this way? □

**6.** If $S$ is a set with $n$ elements, how many different subsets does $S$ have? [Hint: Compare your answers from Problems 4 and 5. Apply your answer from Problem 2.]

*Proof.* The whole homework assignment was setting you up to answer this question. Let $S$ be a set with $n$ elements. You should observe from Problems 4 and 5 that a subset of $S$ is the **same thing** as a function from $S$ to $\{T,F\}$. (The elements *inside* the subset get sent to $T$ and the elements *outside* the subset get sent to $F$.) Therefore the number of subsets of $S$ is the same as the number of functions from $S$ to $\{T,F\}$, which by Problem 2 is

$$\#\{T,F\}^{\#S} = 2^n.$$

We conclude that a set with $n$ elements has exactly $2^n$ different subsets. □

[Remark: When I said in Problem 6 that subsets of $S$ and functions $S \to \{T,F\}$ are the "same thing", what I really meant is that there is a "1-to-1 correspondence" between them. We will discuss the details of this concept later.]