Math 309 **Fall 2014**
Homework 4 Solutions **Drew Armstrong**

Let k and n be integers such that $0 \leq k \leq n$. In this case we define the notation

$$\binom{n}{k} := \frac{n!}{k!\,(n-k)!}.$$

By convention we will say that $\binom{n}{k} := 0$ if $k < 0$ or $k > n$.

1. Consider a standard deck of 52 cards. A subset of 5 cards is called a "hand".

(a) How many different hands are there?
(b) How many different hands are there with all red cards?
(c) How many different hands are there that contain 2 red and 3 black cards?

For part (a), there is a set of 52 cards and we want to choose 5 of them. The total number of possible choices is

$$\binom{52}{5} = \frac{52!}{5!\,47!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot \cancel{47!}}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot \cancel{47!}} = 2598960.$$

That's a lot. For part (b), there is a set of 26 red cards (half the cards are red) and we want to choose 5 of them. The total number of possible choices is

$$\binom{26}{5} = \frac{26!}{5!\,21!} = \frac{26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot \cancel{21!}}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot \cancel{21!}} = 65780.$$

That's not so many. In fact, that seems kind of rare. If we choose 5 cards at random from a standard deck of 52 cards, what is the **probability** that we get all red cards? Well, we haven't discussed probability yet; I'll just tell you that the answer is

$$\frac{65780}{2598960} = \frac{253}{9996} \approx 0.02531.$$

That amounts to a probability of 2.5%. Pretty rare.

For part (c), we want to count the number of different hands that have 2 red cards and 3 black cards. For this purpose we can choose the red and black cards separately and then put them together. There are

$$\binom{26}{2} = \frac{26!}{2!\,24!} = \frac{26 \cdot 25 \cdot \cancel{24!}}{2 \cdot 1 \cdot \cancel{24!}} = \frac{26 \cdot 25}{2} = 325$$

ways to choose 2 red cards from the full set of 26 red cards, and there are

$$\binom{26}{3} = \frac{26!}{3!\,23!} = \frac{26 \cdot 25 \cdot 24 \cdot \cancel{23!}}{3 \cdot 2 \cdot 1 \cdot \cancel{23!}} = \frac{26 \cdot 25 \cdot 24}{3 \cdot 2} = 2600$$

ways to choose 3 black cards from the full set of 26 black cards. Since these two choices can be made independently, the total number of different hands we can make with 2 red and 3 black cards is the product of these two numbers:

$$\binom{26}{2}\binom{26}{3} = 325 \cdot 2600 = 845000.$$

If we choose 5 cards at random from a standard deck of 52, what is the **probability** that we will get 2 red and 3 black cards? Answer:

$$\frac{845000}{2598960} = \frac{1625}{4998} \approx 0.32513.$$

That amounts to a probability of 32.5%. Pretty common.

[Remark: You can tell that I really want to talk about probability, but I'll postpone that discussion until after the second exam.]

2. Vandermonde Convolution. Suppose you have an urn containing R red balls and G green balls. You reach into the urn and grab n balls. Use this situation to give a counting argument for the following identity:

$$\sum_k \binom{R}{k}\binom{G}{n-k} = \binom{R+G}{n}.$$

There are a total of $R+G$ balls in the urn. The total number of ways to choose n of them is just $\binom{R+G}{n}$. **On the other hand**, we could count these choices in a more refined way. Suppose we want to choose k red balls and $n-k$ green balls. The number of ways to choose the red balls is $\binom{R}{k}$ and the number of ways to choose the green balls is $\binom{G}{n-k}$. Thus the total number of choices is the product of these:

$$\binom{R}{k}\binom{G}{n-k}.$$

(Compare with Problem 1(c).) Since every collection of n balls has **some number** of red balls, if we sum the numbers $\binom{R}{k}\binom{G}{n-k}$ over all possible values of k we will recover the total number of choices that we had before:

$$\binom{R+G}{n} = \binom{R}{0}\binom{G}{n} + \binom{R}{1}\binom{G}{n-1} + \cdots + \binom{R}{n-1}\binom{G}{1} + \binom{R}{n}\binom{G}{0}.$$

[Remark: Some of these terms might be zero if $R < n$ or $G < n$.]

The important thing about this problem is that you **believe** the result. For this purpose you should probably do a few examples to convince yourself. You should note, however, that examples are **not** a substitute for a general argument. Here's an example with $R = 2$, $G = 3$, and $n = 3$:

$$\begin{aligned}\binom{5}{2} &= \binom{2}{0}\binom{3}{3} + \binom{2}{1}\binom{3}{2} + \binom{2}{2}\binom{3}{1} + \binom{2}{3}\binom{3}{0}\\ 10 &= 1\cdot 1 + 2\cdot 3 + 1\cdot 3 + 0\cdot 1\\ 10 &= 1 + 6 + 3 + 0.\end{aligned}$$

You should draw the balls $\{r_1, r_2, g_1, g_2, g_3\}$ in an urn and then draw all the sets of three balls, organized by how many red balls each set contains.

3. Use the formula to verify that for relevant values of k and n we have

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

This is just a matter of adding fractions by finding a common denominator, and then hoping that we get the right answer.

$$\begin{aligned}
\binom{n-1}{k} + \binom{n-1}{k-1} &= \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-k)!} \\
&= \frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(k-1)!(n-k)!} \\
&= \frac{(n-k)}{(n-k)} \cdot \frac{(n-1)!}{k!(n-k-1)!} + \frac{k}{k} \cdot \frac{(n-1)!}{(k-1)!(n-k)!} \\
&= \frac{(n-k)(n-1)!}{k!(n-k)!} + \frac{k(n-1)!}{k!(n-k)!} \\
&= \frac{(n-k)(n-1)! + k(n-1)!}{k!(n-k)!} \\
&= \frac{[(n\cancel{-k}) + \cancel{k}](n-1)!}{k!(n-k)!} \\
&= \frac{n(n-1)!}{k!(n-k)!} \\
&= \frac{n!}{k!(n-k)!} \\
&= \binom{n}{k}.
\end{aligned}$$

That's all there is to it.

That's a perfectly good proof, but for me it doesn't really answer the question "why?".

4. Give a **counting argument** for the identity in Problem 3. [Hint: Consider the set of binary strings of length n containing k "1"s. How many are there? Now break this set into two subsets: the strings with leftmost symbol "0" and the strings with leftmost symbol "1". How many are there of each kind?]

Let S be the set of binary strings of length n containing k "1"s and recall that $\#S = \binom{n}{k}$. We can divide this set into two subsets S_0 and S_1, where S_0 is the set of binary strings of length n with containing k "1"s and such that the leftmost bit is "0", and S_1 is the set of binary strings of length n containing k "1"s and such that the leftmost bit is "1". Since the sets S_0 and S_1 are disjoint and exhaust S, we know that

$$\binom{n}{k} = \#S = \#S_0 + \#S_1.$$

[We haven't officially discussed disjoint unions, but I think the equation above is intuitively clear.] Now I claim that $\#S_0 = \binom{n-1}{k}$ and $\#S_1 = \binom{n-1}{k-1}$. Indeed, if the leftmost bit is "0", then the remaining symbols form a binary string of length $n-1$ containing k "1"s, and we know that there are $\binom{n-1}{k}$ of these. Similarly, if the leftmost bit is "1", then the remaining symbols form a binary string of length $n-1$ containing $k-1$ "1"s, and we know that there are $\binom{n-1}{k-1}$ of these. We conclude that

$$\binom{n}{k} = \#S = \#S_0 + \#S_1 = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Again, an example is no substitute for a general argument, but you should always compute an example or two until you feel comfortable with the general argument. Here are the binary strings of length 5 containing 3 "1"s.

11100	01110
11010	01101
11001	01011
10110	00111
10101	
10011	

Note that there are $\binom{5}{3} = 10$ of these, as expected. The strings on the left have leftmost bit "1". Note that there are $\binom{4}{2} = 6$ of these because after stripping away the leftmost "1" we are left with the binary strings of length 4 containing 2 "1"s:

$$1100, \quad 1010, \quad 1001, \quad 0110, \quad 0101, \quad 0011.$$

The strings on the right have leftmost bit "0". Note that there are $\binom{4}{3} = 4$ of these because after stripping away the leftmost "0" we are left with the binary strings of length 4 containin 3 "0"s:

$$1110, \quad 1101, \quad 1011, \quad 0111.$$

This explains **why** $\binom{5}{3} = \binom{4}{3} + \binom{4}{2}$.

5. Trinomial Coefficients. Consider integers $i, j, k \geq 0$ such that $i + j + k = n$. Let N be the number of different words of length n containing i "a"s, j "b"s, and k "c"s. Explain why

$$n! = N \cdot i! \cdot j! \cdot k!$$

[Hint: Count the permutations of the symbols $a_1, \ldots, a_i, b_1, \ldots, b_k, c_1, \ldots, c_j$ in two different ways.] Use the result to compute the number of different words (not necessarily English words) that can be made from the letters

$$b, a, n, a, n, a.$$

Let $i, j, k \geq 0$ be nonnegative integers such that $i + j + k = n$, and let N be the number of words of length n containing i "a"s, j "b"s, and k "c"s. We want to find a formula for the number N. To do this we will instead solve a slightly different problem: We will count the number of words formed from the **labeled** letters

$$a_1, \ldots, a_i, b_1, \ldots, b_k, c_1, \ldots, c_j.$$

in **two different ways**. On one hand, these are just n different symbols, so the number words they can form is $n!$. On the other hand, we could form such a word by **first** choosing an **unlabeled** word in N ways, and **then** placing labels on the "a"s in $i!$ ways, on the "b"s in $j!$ ways, and on the "c"s in $k!$ ways. Since we have counted the same objects twice, we get an equality

$$n! = N \cdot i! \cdot j! \cdot k!$$

which we can solve to obtain

$$\boxed{N = \frac{n!}{i!j!k!}.}$$

So how many words can you make using all of the letters: b, a, n, a, n, a. The answer is

$$\frac{6!}{1!2!3!} = \frac{6 \cdot 5 \cdot 4 \cdot \cancel{3 \cdot 2 \cdot 1}}{1 \cdot 2 \cdot 1 \cdot \cancel{3 \cdot 2 \cdot 1}} = \frac{6 \cdot 5 \cdot 4}{2} = 60.$$