Math 309 — Fall 2014
Homework 3 Solutions — Drew Armstrong

On this homework you will meet some new Boolean functions.

1. Given $P, Q \in \{T, F\}$ we define the **Boolean sum** (also called "exclusive OR"):

$$P \oplus Q := (P \wedge \neg Q) \vee (\neg P \wedge Q).$$

(a) Draw the truth table for $P \oplus Q$.
(b) Use truth tables to prove that for all $P, Q, R \in \{T, F\}$ we have

$$P \wedge (Q \oplus R) = (P \wedge Q) \oplus (P \wedge R).$$

[It is fair to think of $\oplus$ as "addition" and $\wedge$ as "multiplication".]

For part (a) we have the following truth table.

P	Q	$\neg P$	$\neg Q$	$P \wedge \neg Q$	$\neg P \wedge Q$	$(P \wedge \neg Q) \vee (\neg P \wedge Q)$
T	T	F	F	F	F	F
T	F	F	T	T	F	T
F	T	T	F	F	T	T
F	F	T	T	F	F	F

The last column represents $P \oplus Q$. We could also have jumped right to the answer because $P \oplus Q$ was given to us in disjunctive normal form (which is equivalent to just describing the truth table).

For part (b) we have the following truth table.

P	Q	R	$Q \oplus R$	$P \wedge (Q \oplus R)$	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \oplus (P \wedge R)$
T	T	T	F	F	T	T	F
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	F	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Note that the fifth and eighth columns are the same.

[There is another way to think about part (b) if you know something about modular arithmetic. If we let $T = 1$ and $F = 0$ then the operation $\wedge$ is the same as "multiplication mod 2" and the operation $\oplus$ is the same as "addition mod 2".

$\wedge$	1	0
1	1	0
0	0	0

$\oplus$	1	0
1	0	1
0	1	0

In this language the identity $P \wedge (Q \oplus R) = (P \wedge Q) \oplus (P \wedge R)$ is just the usual distributivity of multiplication over addition.]

2. Given $P, Q \in \{T, F\}$ we define the function $P \Rightarrow Q$ with the following table:

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

We call this **logical implication** and we read $P \Rightarrow Q$ as "if P then Q" or "P implies Q".

(a) Draw the truth table for $P \not\Rightarrow Q := \neg(P \Rightarrow Q)$.
(b) Compute the disjunctive normal form of $P \not\Rightarrow Q$.
(c) Use part (b) to find a simple formula for $P \Rightarrow Q$. [Hint: De Morgan's Law.]

For part (a) we have the following truth table.

P	Q	$P \Rightarrow Q$	$P \not\Rightarrow Q$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

For part (b) we note that the disjunctive normal form of $P \not\Rightarrow Q$ is just

$$P \not\Rightarrow Q = P \wedge \neg Q$$

where the term $P \wedge \neg Q$ corresponds to the single T in the truth table for $P \neg \Rightarrow Q$.

For part (c), let me first note that the disjunctive normal form of $P \Rightarrow Q$ is

$$P \Rightarrow Q = (P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q),$$

which is not very simple. We can get a nicer formula if we start with our formula for $P \not\Rightarrow Q$ and then apply de Morgan's law:

$$\begin{aligned} P \Rightarrow Q &= \neg(P \not\Rightarrow Q) \\ &= \neg(P \wedge \neg Q) \\ &= \neg P \vee \neg\neg Q \\ &= \neg P \vee Q. \end{aligned}$$

That's better.

3. For all $P, Q \in \{T, F\}$ we define the function $P \Leftrightarrow Q$ by

$$P \Leftrightarrow Q := (P \Rightarrow Q) \wedge (Q \Rightarrow P).$$

We call this function **logical equivalence** and we read $P \Leftrightarrow Q$ as "P if and only if Q".

(a) Compute the disjunctive normal form of $P \Leftrightarrow Q$.
(b) Show that $P \not\Leftrightarrow Q := \neg(P \Leftrightarrow Q)$ is the same as $P \oplus Q$.

For part (a) we first compute the truth table of $P \Leftrightarrow Q$ as follows.

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

(Observe that $\Leftrightarrow$ acts just like an equals sign; it returns T if the arguments are the same and it returns F if the arguments are different.) Now we can read the disjunctive normal form directly from the truth table:

$$P \Leftrightarrow Q = (P \wedge Q) \vee (\neg P \wedge \neg Q).$$

There is not much to do for part (b). We just draw the truth table and observe that the fourth and fifth columns are the same.

P	Q	$P \Leftrightarrow Q$	$P \not\Leftrightarrow Q$	$P \oplus Q$
T	T	T	F	F
T	F	F	T	T
F	T	F	T	T
F	F	T	F	F

Now we have three different ways to think about the operation $\oplus$. It can be a Boolean analogue of "addition", it can be the "exclusive or" logical operation, and we can also think of it as "not equal to".

4. Let B be a Boolean algebra. For all $P, Q \in B$ we define the "Sheffer stroke"

$$P \uparrow Q := \neg(P \wedge Q).$$

Use the properties of Boolean algebra from the handout to prove the following formulas. Don't use truth tables! These formulas can be used to express **any** function $\{T, F\}^n \to \{T, F\}$ in terms of $\uparrow$ alone.

(a) $\neg P = P \uparrow P$
(b) $P \vee Q = (P \uparrow P) \uparrow (Q \uparrow Q)$
(c) $P \wedge Q = (P \uparrow Q) \uparrow (P \uparrow Q)$

In this problem we will avoid truth tables and instead use synthetic Boolean algebra. I will write each part as a two-line proof, quoting axioms and theorems using their number from the handout. For part (a) we have

$$\begin{aligned} P \uparrow P &= \neg(P \wedge P) && \text{by definition} \\ &= \neg P. && (6) \end{aligned}$$

For part (b) we have

$$\begin{aligned} (P \uparrow P) \uparrow (Q \uparrow Q) &= \neg P \uparrow \neg Q && \text{by part (a)} \\ &= \neg(\neg P \wedge \neg Q) && \text{by definition} \\ &= \neg\neg P \vee \neg\neg Q && (12) \\ &= P \vee Q. && ? \end{aligned}$$

OOPS. We never proved that $\neg\neg P = P$ did we? Let's prove it now. We will use Theorem 11 (Uniqueness of Complements). To do this we note that

$$\begin{aligned} \neg P \wedge P &= P \wedge \neg P && (2) \\ &= 0, && (4) \end{aligned}$$

and

$$\begin{aligned} \neg P \vee P &= P \vee \neg P && (2) \\ &= 1, && (4) \end{aligned}$$

Then by (11) we conclude that P must equal the complement of $\neg P$. In other words, $P = \neg\neg P$. Let's call this Theorem (13). This completes our proof of part (b). [Don't worry if you didn't fill in this last detail. You won't lose any points for that.]

Finally, for part (c) we have

$$\begin{aligned}
(P \uparrow Q) \uparrow (P \uparrow Q) &= \neg((P \uparrow Q) \wedge (P \uparrow Q)) && \text{by definition} \\
&= \neg(P \uparrow Q) && (6) \\
&= \neg(\neg(P \wedge Q)) && \text{by definition} \\
&= P \wedge Q && (13).
\end{aligned}$$

That's it.

[The results of Problem 4 prove that the Sheffer stroke is "universal". This means that we can express **any** Boolean function using just the Sheffer stroke. For example, consider the function

$$\varphi(P, Q, R) = (P \wedge \neg Q) \vee R.$$

Then we have

$$\begin{aligned}
\varphi(P, Q, R) &= (P \wedge \neg Q) \vee R \\
&= (P \wedge (Q \uparrow Q)) \vee R \\
&= ((P \wedge (Q \uparrow Q)) \uparrow (P \wedge (Q \uparrow Q)) \uparrow (R \uparrow R) \\
&= (((P \uparrow (Q \uparrow Q)) \uparrow (P \uparrow (Q \uparrow Q))) \uparrow ((P \uparrow (Q \uparrow Q)) \uparrow (P \uparrow (Q \uparrow Q))) \uparrow (R \uparrow R)
\end{aligned}$$

Obviously this is not a good language for humans to use, but computers are quite happy with it. In fact, this is the language that is used inside of flash memory drives.]