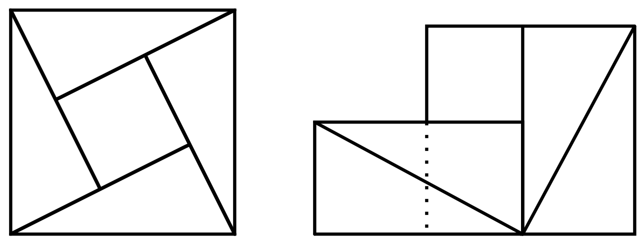
**Math 230** **Fall 2018**
**Homework 1** **Drew Armstrong**

---

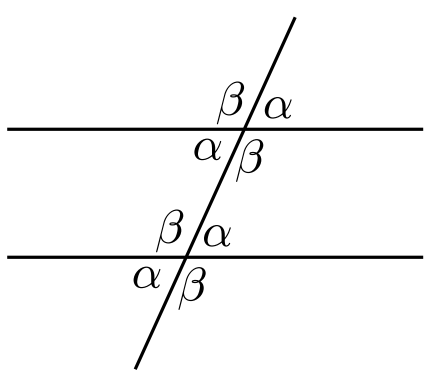
**Problem 1.** In the *Lilavati*, the Indian mathematician Bhaskara (1114–1185) gave a one-word proof of the Pythagorean theorem. He said: **"Behold!"**



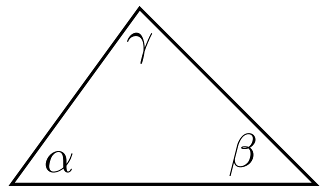
**Add words to the proof.** Your goal is to persuade a high school student who claims that he/she doesn't "get it." Try to avoid algebra as much as possible.

*Proof.* Consider a right triangle with side lengths $a, b, c$ in increasing order, and consider the two figures above. There are three important observations: **(1)** The two figures have the same area because each is made of the same five puzzle pieces, i.e., four copies of the right triangle and one square of side length $b-a$. **(2)** The figure on the left has area $c^2$ because it is a square of side length $c$. **(3)** The dotted line in the right figure divides it into two squares of sides lengths $a$ and $b$, respectively. Hence this figure has area $a^2+b^2$. (You might need to put labels on the figure if your high school student is skeptical.) By combining the observations **(1)**, **(2)**, **(3)** we conclude that $c^2 = a^2 + b^2$. □

**Problem 2.** Prove that the interior angles of any triangle sum to 180°. You may use the following two facts without proof. **Prop I.31:** Given a line $\ell$ and a point $p$ not on $\ell$, **it is possible** to draw a line through $p$ parallel to $\ell$. **Prop I.29:** If a line falls on two parallel lines, then the corresponding angles are equal, as in the following figure.

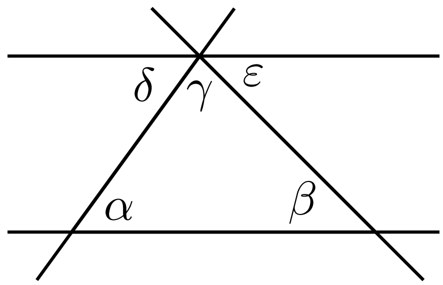



*Proof.* Consider an arbitrary triangle with interior angles $\alpha$, $\beta$, and $\gamma$:




We will show that $\alpha + \beta + \gamma = 180°$.

To do this, we first use Proposition I.31 to draw a line through the vertex at angle $\gamma$ that is **parallel** to the opposite side of the triangle. Then after extending the three sides of the triangle we obtain the following diagram:




The labeled angles $\delta$ and $\varepsilon$ are initially unknown to us. However, by applying Proposition I.29 twice we find that $\delta = \alpha$ and $\varepsilon = \beta$. Since $\delta$, $\gamma$, and $\varepsilon$ form a straight line we must have $\delta + \gamma + \varepsilon = 180°$. Finally, we conclude that

$$\alpha + \beta + \gamma = \delta + \varepsilon + \gamma = 180°,$$

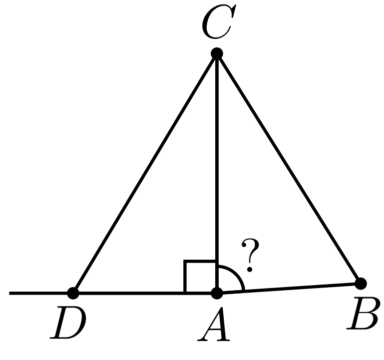
as desired. □

**Problem 3.** Look up Euclid's Proposition I.48. Tell me the statement and the proof. Your goal is to make everything as understandable as possible, especially to yourself but also to me. Imagine that you are trying to teach this to someone.

**Proposition I.48.** Consider a triangle with vertices $A$, $B$, and $C$. If the side lengths satisfy $\overline{AB}^2 + \overline{AC}^2 = \overline{BC}^2$, then I claim that the angle $\measuredangle BAC$ is right.

*Proof.* Let us **assume** that $\overline{AB}^2 + \overline{AC}^2 = \overline{BC}^2$. In this case we will prove that $\measuredangle BAC$ is a right angle.

First we draw a line through $A$ perpendicular to the line $AC$ [using Prop I.11]. Then we find the point $D$ on this line such that $\overline{AD} = \overline{AB}$ [using Prop I.3]. After connecting the line segment $DC$ [using Postulate 1] we obtain the following diagram, where the angle $\measuredangle BAC$ is unknown to us:



Since $\overline{DA} = \overline{AB}$ [by construction] we must have $\overline{DA}^2 = \overline{AB}^2$. Then since the angle $\measuredangle DAC$ is right [by construction] we can apply the Pythagorean Theorem [Prop I.47] to get

$$\begin{aligned} \overline{DC}^2 &= \overline{DA}^2 + \overline{AC}^2 && \text{[Prop I.47]} \\ &= \overline{AB}^2 + \overline{AC}^2 && \text{[CN 2]} \\ &= \overline{BC}^2 && \text{[by assumption]} \end{aligned}$$

We conclude [by CN1] that $\overline{DC}^2 = \overline{BC}^2$, and hence $\overline{DC} = \overline{BC}$. Now the triangles $\triangle DAC$ and $\triangle BAC$ have all of the their corresponding sides of equal length, so the triangles are congruent [by Prop I.8, i.e., the "side-side-side" criterion for triangle congruence]. By definition of congruence this means that the triangles also have the same angles. In particular, the angle $\measuredangle DAC$ equals the angle $\measuredangle BAC$. Since angle $\measuredangle DAC$ is right [by construction], we conclude that the angle $\measuredangle BAC$ is also right. □

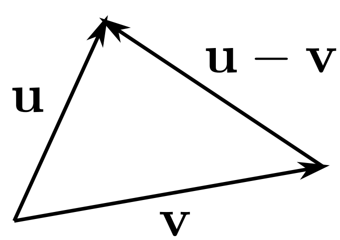
**Problem 4.** The dot product of the vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ is defined by $\mathbf{u} \bullet \mathbf{v} := u_1v_1 + u_2v_2$. The length $\|\mathbf{u}\|$ of a vector $\mathbf{u}$ satisfies $\|\mathbf{u}\|^2 = \mathbf{u} \bullet \mathbf{u}$.

(a) The vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} - \mathbf{v}$ form the three sides of a triangle. Draw this triangle.
(b) Use algebra (not geometry) to prove that $\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2(\mathbf{u} \bullet \mathbf{v})$.
(c) Use the formula from part (b) to prove the following statement:

"the vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if and only if $\mathbf{u} \bullet \mathbf{v} = 0$."

[Hint: Remember your picture from part (a). What do the Pythagorean Theorem and its converse tell you about this picture?]

The triangle for part (a) looks like this:




(Since we discussed this in class, I imagine everyone's picture will look roughly the same.) Part (b) is simply a computation:

$$\begin{aligned} \|\mathbf{u} - \mathbf{v}\|^2 &= \|(u_1, u_2) - (v_1, v_2)\|^2 \\ &= \|(u_1 - v_1, u_2 - v_2)\|^2 \\ &= (u_1 - v_1)^2 + (u_2 - v_2)^2 \\ &= (u_1^2 - 2u_1v_1 + v_1^2) + (u_2^2 - 2u_2v_2 + v_2^2) \\ &= (u_1^2 + u_2^2) + (v_1^2 + v_2)^2 - 2(u_1v_1 + u_2v_2) \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2(\mathbf{u} \bullet \mathbf{v}). \end{aligned}$$

Part (c) asks for a proof, so I'll write it nicely.

*Proof.* Consider any two vectors $\mathbf{u} = (u_1, v_1)$ and $\mathbf{v} = (v_1, v_2)$. We will prove that $\mathbf{u}$ and $\mathbf{v}$ are perpendicular (i.e., $\mathbf{u} \perp \mathbf{v}$) if and only if $\mathbf{u} \bullet \mathbf{v} = u_1v_1 + u_2v_2 = 0$. Since this is an "if and only if" statement we must prove each direction separately.

First we will prove that $\mathbf{u} \perp \mathbf{v}$ implies $\mathbf{u} \bullet \mathbf{v} = 0$. So let us assume that $\mathbf{u} \perp \mathbf{v}$. In this case, the triangle from part (a) is a right triangle and the Pythagorean Theorem tells us that

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

On the other hand, we know from part (b) that

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2(\mathbf{u} \bullet \mathbf{v}).$$

Equating these two expressions for $\|\mathbf{u} - \mathbf{v}\|^2$ gives

$$\begin{aligned} \cancel{\|\mathbf{u}\|^2} + \cancel{\|\mathbf{v}\|^2} &= \cancel{\|\mathbf{u}\|^2} + \cancel{\|\mathbf{v}\|^2} - 2(\mathbf{u} \bullet \mathbf{v}) \\ 0 &= -2(\mathbf{u} \bullet \mathbf{v}) \\ 0 &= \mathbf{u} \bullet \mathbf{v}, \end{aligned}$$

as desired.

Now we will prove that $\mathbf{u} \bullet \mathbf{v} = 0$ implies $\mathbf{u} \perp \mathbf{v}$. So let us assume that $\mathbf{u} \bullet \mathbf{v} = 0$. Then from part (b) we have

$$\begin{aligned} \|\mathbf{u} - \mathbf{v}\|^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2(\mathbf{u} \bullet \mathbf{v}) \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2(0) \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2. \end{aligned}$$

Finally, by applying this fact to the triangle in part (a) we conclude from the **converse** of the Pythagorean Theorem that the angle between $\mathbf{u}$ and $\mathbf{v}$ is $90°$. In other words, we conclude that $\mathbf{u} \perp \mathbf{v}$. □

[Remark: Of course we could have proved this result more quickly by quoting the theorem

$$\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \cdot \cos\theta,$$

but that would violate one of the main principles of proof writing: don't prove a simple result by quoting a more complicated result. To do so would put you at risk of circular reasoning.]