Math 230 D **Fall 2015**
Exam 2 — Fri Oct 30 **Drew Armstrong**

There are 4 problems, worth 6 points each. This is a closed book test. Anyone caught cheating will receive a score of **zero**.

Problem 1. Division Theorem.

(a) Accurately state the Division Theorem.

(b) Use the Division Theorem to prove that there is **no** integer $n \in \mathbb{Z}$ satisfying the property $3n = 4$.

Problem 2. Axioms of $\mathbb{Z}$. Consider the following three axioms:

(1) $\forall a, b, c \in \mathbb{Z}$, $a + (b + c) = (a + b) + c$.
(2) $\exists 0 \in \mathbb{Z}$, $\forall a \in \mathbb{Z}$, $a + 0 = 0 + a = a$.
(3) $\forall a \in \mathbb{Z}$, $\exists - a \in \mathbb{Z}$, $a + (-a) = (-a) + a = 0$.

(a) Explicitly use the axioms to prove the following Cancellation Lemma:

$$\forall a, b, c \in \mathbb{Z}, (a + b = a + c) \Rightarrow (b = c).$$

(b) Let $a \in \mathbb{Z}$ and suppose there exists $a' \in \mathbb{Z}$ such that $a + a' = 0$. In this case prove that $a' = (-a)$. [Hint: You can quote the Cancellation Lemma from part (a).]

Problem 3. Linear Diophantine Equations.

(a) Use the Extended Euclidean Algorithm to find **one particular solution** $x', y' \in \mathbb{Z}$ to the equation $22x' + 16y' = 2$.

(b) Write down the **complete solution** $x, y \in \mathbb{Z}$ to the equation $22x + 16y = 0$.

(c) Write down the **complete solution** $x, y \in \mathbb{Z}$ to the equation $22x + 16y = 2$.

Problem 4. Well-Ordering.

(a) Accurately state some version of the Well-Ordering Axiom.

(b) Use Well-Ordering to prove that **every** integer $n > 1$ is divisible by a prime number. [Hint: Assume for contradiction that **there exists** an integer $n > 1$ such that n is **not** divisible by a prime number.]

Math 230 E **Fall 2013**
Exam 2 — Fri Nov 8 **Drew Armstrong**

There are 4 problems, worth 5 points each. This is a closed book test. Anyone caught cheating will receive a score of **zero**.

Problem 1.

(a) Use the Euclidean Algorithm to show that $\gcd(41, 12) = 1$.

(b) Use the Extended Euclidean Algorithm to find **one specific solution** $x, y \in \mathbb{Z}$ to the equation $41x + 12y = 1$.

(c) Tell me **infinitely many solutions** $x, y \in \mathbb{Z}$ to the equation $41x + 12y = 1$. [You don't need to find all of them.]

Problem 2. Fix a nonzero integer $0 \neq n$ and define a relation $\equiv_n$ on $\mathbb{Z}$ as follows:

$$a \equiv_n b \qquad \Longleftrightarrow \qquad n|(a-b)$$

(a) For all $a \in \mathbb{Z}$ prove that $a \equiv_n a$.

(b) For all $a, b \in \mathbb{Z}$ prove that $(a \equiv_n b) \Rightarrow (b \equiv_n a)$.

(c) For all $a, b, c \in \mathbb{Z}$ prove that $(a \equiv_n b \text{ AND } b \equiv_n c) \Rightarrow (a \equiv_n c)$.

Problem 3. Let $a, b \in \mathbb{Z}$ and $d = \gcd(a, b)$.

(a) Accurately state Bézout's Identity.

(b) If $a = da'$ and $b = db'$, prove that there exist $x, y \in \mathbb{Z}$ such that $1 = a'x + b'y$.

(c) Use part (b) to prove that $\gcd(a', b') = 1$.

Problem 4. Consider a sequence of integers $n_1, n_2, n_3, \ldots \mathbb{Z}$ such that

$$n_1 > n_2 > n_3 > \cdots \geq 0.$$

You will prove that there exists some k such that $n_k = 0$.

(a) Accurately state the Well-Ordering Axiom.

(b) Assume for contradiction that no such k exists and consider the set $S = \{n_1, n_2, \ldots\}$. What does the Well-Ordering Axiom say about this set?

(c) Use part (b) to derive a contradiction.