Math 230 D **Fall 2015**
Homework 2 **Drew Armstrong**

Problem 1. Logical Analysis.

(a) Let Q and R be logical statements. Use a truth table to prove that $\neg(Q \vee R)$ is logically equivalent to $\neg Q \wedge \neg R$. [This is called de Morgan's law.]

(b) Let P, Q, and R be logical statements. Use a truth table to prove that $(Q \vee R) \Rightarrow P$ is logically equivalent to $(Q \Rightarrow P) \wedge (R \Rightarrow P)$.

(c) Apply the principles from (a) and (b) to prove that for all integers m and n we have

$$\text{"}mn \text{ is even"} \iff \text{"}m \text{ is even or } n \text{ is even"}.$$

[Hint: Let $P =$"mn is even", $Q =$"m is even", and $R =$"n is even". Use part (a) for the "$\Rightarrow$" direction and use part (b) for the "$\Leftarrow$" direction.]

Problem 2. Absolute Value. Given an integer a we define its absolute value as follows:

$$|a| := \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -a & \text{if } a < 0 \end{cases}$$

Prove that for all integers a and b we have $|ab| = |a||b|$. [Hint: Your proof will break into at least five separate cases. You may assume without proof the properties $(-a)(-b) = ab$ and $(-a)b = a(-b) = -(ab)$; we'll prove them later.]

Problem 3. Divisibility. Given integers m and n we will write "$m|n$" to mean that "there exists an integer k such that $n = mk$" and when this is the case we will say that "m divides n" or "n is divisible by m". Now let a, b, and c be integers. Prove the following properties.

(a) If $a|b$ and $b|c$ then $a|c$.

(b) If $a|b$ and $a|c$ then $a|(bx + cy)$ for all integers x and y.

(c) If $a|b$ and $b|a$ then $a = \pm b$. [Hint: Use the fact that $uv = 0$ implies $u = 0$ or $v = 0$.]

(d) If $a|b$ and b is nonzero then $|a| \leq |b|$. [Hint: Use the result of Problem 2.]

Problem 4. The Square Root of 5. Prove that $\sqrt{5}$ is not a ratio of integers, in two steps.

(a) First prove the following **lemma**: Let n be an integer. If n^2 is divisible by 5, then so is n. [Hint: Use the contrapositive and note that there are four separate ways for an integer to be **not** divisible by 5. Sorry it's a bit tedious; we will find a better way to do this later.]

(b) Use the method of contradiction to prove that $\sqrt{5}$ is not a ratio of integers. Explicitly quote your lemma in the proof. [Hint: Your proof should begin as follows: "Assume for contradiction that $\sqrt{5}$ **is** a ratio of integers. In this case, ..."]