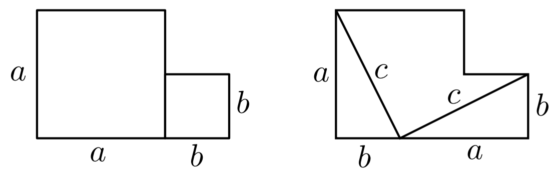
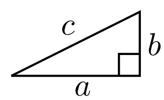
Math 230 D **Fall 2015**
Homework 1 **Drew Armstrong**

Problem 1. In this problem you will prove the Pythagorean Theorem by cutting up the same shape in two different ways.

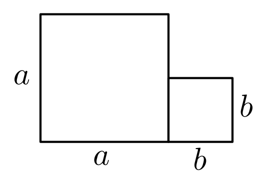


[Hint: The three pieces of the right diagram can be rearranged into a square. Your proof should begin like this: "Consider a right angled triangle with side lengths a, b, and c, where c is is the hypotenuse. (Now draw a picture of the triangle.) In this case we will prove that $a^2 + b^2 = c^2$. To do this we will cut up the same shape in two different ways. Consider the following two diagrams…"]

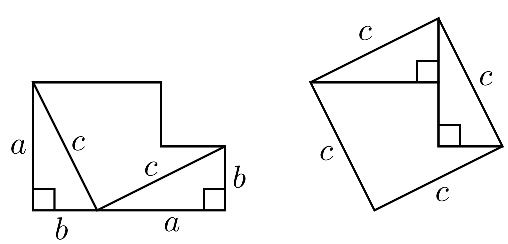
Proof. Consider a right angled triangle with side lengths a, b, and c, where c is the hypotenuse:



In this case we will prove that $a^2 + b^2 = c^2$. To do this we will cut up the same shape in two different ways. Consider the shape obtained by gluing together a square of side length a with a square of side length b:



Clearly this shape has area $a^2 + b^2$. On the other hand, we can cut two copies of the original triangle from this shape and rearrange the three pieces into a square of area c^2:



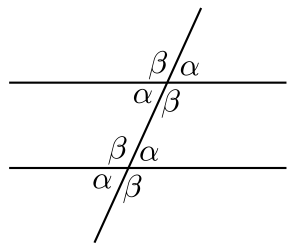
Since cutting up and rearraging the pieces doesn't change the area, we conclude that the original shape must also have had area c^2. Since the area of the original shape was $a^2 + b^2$, we conclude that

$$a^2 + b^2 = c^2.$$

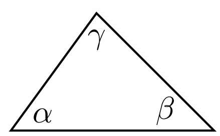
□

[That was the main outline of the proof. Depending on the audience, we might want to dive into a few details to convince them that the three rearranged pieces really do fit together to form a square. (Remember, diagrams can be deceiving.) To do this we will need to use the fact that the three interior angles of a triangle add to $180°$. If our audience isn't willing to believe that fact, then we will show them our argument from Problem 2.]

Problem 2. Prove that the interior angles of any triangle sum to $180°$. You may use the following two facts without proof. **Prop I.31:** Given a line ℓ and a point p not on ℓ, **it is possible** to draw a line through p parallel to ℓ. **Prop I.29:** If a line falls on two parallel lines, then the corresponding angles are equal, as in the following figure.

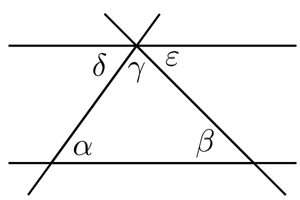


Proof. Consider an arbitrary triangle with interior angles α, β, and γ:



We will show that $\alpha + \beta + \gamma = 180°$.

To do this, we first use Proposition I.31 to draw a line through the vertex at angle γ that is **parallel** to the opposite side of the triangle. Then after extending the three sides of the triangle we obtain the following diagram:



The labeled angles δ and ε are initially unknown to us. However, by applying Proposition I.29 twice we find that $\delta = \alpha$ and $\varepsilon = \beta$. Since δ, γ, and ε form a straight line we must have $\delta + \gamma + \varepsilon = 180°$. Finally, we conclude that

$$\begin{aligned} \alpha + \beta + \gamma &= \delta + \varepsilon + \gamma \\ &= 180°, \end{aligned}$$

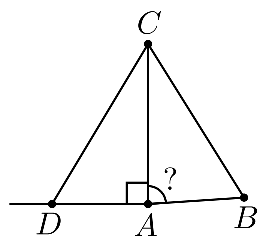
as desired. □

Problem 3. Look up Euclid's Proposition I.48. Tell me the statement and the proof. Your goal is to make everything as understandable as possible, especially to yourself but also to me. Imagine that you are trying to teach this to someone.

Proposition I.48. Consider a triangle with vertices A, B, and C. If the side lengths satisfy $\overline{AB}^2 + \overline{AC}^2 = \overline{BC}^2$, then I claim that the angle BAC is right.

Proof. Let us **assume** that $\overline{AB}^2 + \overline{AC}^2 = \overline{BC}^2$. In this case we will prove that BAC is a right angle.

First we draw a line through A perpendicular to the line AC [using Prop I.11]. Then we find the point D on this line such that $\overline{AD} = \overline{AB}$ [using Prop I.3]. After connecting the line segment DC [using Postulate 1] we obtain the following diagram:



Since $\overline{DA} = \overline{AB}$ [by construction] we must have $\overline{DA}^2 = \overline{AB}^2$. Since the angle DAC is right [by construction] we can apply the Pythagorean Theorem [Prop I.47] to get

$$\begin{aligned}
\overline{DC}^2 &= \overline{DA}^2 + \overline{AC}^2 && \text{[Prop I.47]} \\
&= \overline{AB}^2 + \overline{AC}^2 && \text{[CN 2]} \\
&= \overline{BC}^2 && \text{[by assumption]}
\end{aligned}$$

We conclude [by CN1] that $\overline{DC}^2 = \overline{BC}^2$, and hence $\overline{DC} = \overline{BC}$. Now the triangles DAC and BAC have all of the their corresponding sides of equal length, so the triangles are congruent [by Prop I.8, i.e., the "side-side-side" criterion for triangle congruence]. In particular, the angle DAC equals the angle BAC. Since angle DAC is right [by construction], we condlude that the angle BAC is right, as desired. □

Problem 4. The dot product of the vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ is defined by $\mathbf{u} \bullet \mathbf{v} := u_1 v_1 + u_2 v_2$. The length $\|\mathbf{u}\|$ of a vector $\mathbf{u}$ satisfies $\|\mathbf{u}\|^2 = \mathbf{u} \bullet \mathbf{u}$.

(a) The vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} - \mathbf{v}$ form the three sides of a triangle. Draw this triangle.

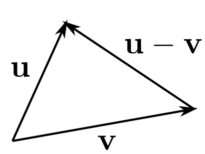
(b) Use algebra (not geometry) to prove that $\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2(\mathbf{u} \bullet \mathbf{v})$.

(c) Use the formula from part (b) to prove the following statement:

"the vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if and only if $\mathbf{u} \bullet \mathbf{v} = 0$."

[Hint: Remember your picture from part (a). What do the Pythagorean Theorem and its converse tell you about this picture?]

The triangle for part (a) looks like this:



(Since we discussed this in class, I imagine everyone's picture will look roughly the same.) Part (b) is simply a computation:

$$\begin{aligned}
\|\mathbf{u} - \mathbf{v}\|^2 &= \|(u_1, u_2) - (v_1, v_2)\|^2 \\
&= \|(u_1 - v_1, u_2 - v_2)\|^2 \\
&= (u_1 - v_1)^2 + (u_2 - v_2)^2
\end{aligned}$$

$$\begin{aligned}
&= (u_1^2 - 2u_1v_1 + v_1^2) + (u_2^2 - 2u_2v_2 + v_2^2) \\
&= (u_1^2 + u_2^2) + (v_1^2 + v_2)^2 - 2(u_1v_1 + u_2v_2) \\
&= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2(\mathbf{u} \bullet \mathbf{v}).
\end{aligned}$$

Part (c) asks for a proof, so I'll write it nicely.

Proof. Consider any two vectors $\mathbf{u} = (u_1, v_1)$ and $\mathbf{v} = (v_1, v_2)$. We will prove that $\mathbf{u}$ and $\mathbf{v}$ are perpendicular (i.e., $\mathbf{u} \perp \mathbf{v}$) if and only if $\mathbf{u} \bullet \mathbf{v} = u_1v_1 + u_2v_2 = 0$. Since this is an "if and only if" statement, we will prove both directions separately.

First we will prove that $\mathbf{u} \perp \mathbf{v}$ implies $\mathbf{u} \bullet \mathbf{v} = 0$. So assume that $\mathbf{u} \perp \mathbf{v}$. In this case, the triangle from part (a) is a right triangle and the Pythagorean Theorem tells us that

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

On the other hand, we know from part (b) that

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2(\mathbf{u} \bullet \mathbf{v}).$$

Equating the right hand sides gives

$$\begin{aligned}
\cancel{\|\mathbf{u}\|^2} + \cancel{\|\mathbf{v}\|^2} &= \cancel{\|\mathbf{u}\|^2} + \cancel{\|\mathbf{v}\|^2} - 2(\mathbf{u} \bullet \mathbf{v}) \\
0 &= -2(\mathbf{u} \bullet \mathbf{v}) \\
0 &= \mathbf{u} \bullet \mathbf{v},
\end{aligned}$$

as desired.

Next we will prove that $\mathbf{u} \bullet \mathbf{v} = 0$ implies $\mathbf{u} \perp \mathbf{v}$. So assume that $\mathbf{u} \bullet \mathbf{v} = 0$. Then from part (b) we have

$$\begin{aligned}
\|\mathbf{u} - \mathbf{v}\|^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2(\mathbf{u} \bullet \mathbf{v}) \\
&= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2(0) \\
&= \|\mathbf{u}\|^2 + \|\mathbf{v}|^2.
\end{aligned}$$

If we apply this fact to the triangle in part (a), then the **converse** of the Pythagorean Theorem tells us that the angle between $\mathbf{u}$ and $\mathbf{v}$ is $90°$. In other words we have $\mathbf{u} \perp \mathbf{v}$, as desired. $\square$