Math 230 D **Fall 2015**
Exam 1 — Fri Sept 25 **Drew Armstrong**

There are 4 problems, worth 6 points each. This is a closed book test. Anyone caught cheating will receive a score of **zero**.

Problem 1.

(a) State the principle of the contrapositive.

Let P and Q be logical statements. Then the statement $P \Rightarrow Q$ is logically equivalent to the statement $\neg Q \Rightarrow \neg P$.

(b) State De Morgan's law.

Let P and Q be logical statements. Then we have
- $\neg(P \vee Q) = (\neg P \wedge \neg Q)$
- $\neg(P \wedge Q) = (\neg P \vee \neg Q)$

(c) Tell me the **opposite** of the following statement: "Every even number greater than 2 is a sum of two prime numbers."

Let me put the key words in boldface: "**Every** even number greater than 2 **is** a sum of two prime numbers." If S is the set of even numbers greater than 2 and if $P(n)$ is the statment "n is a sum of two prime numbers", then our statement can be written as

$$\forall n \in S, P(n).$$

The opposite statement is

$$\neg(\forall n \in S, P(n)) = (\exists n \in S, \neg P(n)).$$

In other words: "**There exists** an even number greater than 2 that is **not** a sum of two prime numbers."

[Remark: The intermediate work was not necessary. I awarded full points if you just skipped to the answer. The statement "$\forall n \in S, P(n)$" is called "Goldbach's conjecture". People believe it is true but nobody knows how to prove it.]

Problem 2.

(a) Draw the truth table for the Boolean function $P \vee Q$.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

(b) Use a truth table to prove that $P \vee Q$ is logically equivalent to $(P \Rightarrow Q) \Rightarrow Q$.

P	Q	$P \vee Q$	$P \Rightarrow Q$	$(P \Rightarrow Q) \Rightarrow Q$
T	T	T	T	T
T	F	T	F	T
F	T	T	T	T
F	F	F	T	F

(c) Now use the result from part (b) together with De Morgan's law to express $P \wedge Q$ using **only** the functions $\neg$ and $\Rightarrow$ (i.e., don't use $\vee$ or $\wedge$).

Applying the result of part (b) to the statements $\neg P$ and $\neg Q$ gives

$$(\neg P \vee \neg Q) = ((\neg P \Rightarrow \neg Q) \Rightarrow \neg Q).$$

and applying De Morgan's law gives

$$\begin{aligned} \neg(P \wedge Q) &= (\neg P \vee \neg Q) \\ (P \wedge Q) &= \neg(\neg P \vee \neg Q). \end{aligned}$$

Putting these two equations together gives

$$(P \wedge Q) = \neg((\neg P \Rightarrow \neg Q) \Rightarrow \neg Q).$$

This can be simplified slightly, but not much.

Problem 3.

(a) Consider two real numbers $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq 0$. Use the method of contradiction to prove the following statement: "if $\alpha \in \mathbb{Q}$ and $\beta \notin \mathbb{Q}$ then $\alpha\beta \notin \mathbb{Q}$".
[Hint: Let $\alpha \in \mathbb{Q}$ and $\beta \notin \mathbb{Q}$. Now assume for contradiction that $\alpha\beta \in \mathbb{Q}$.]

Proof. Let $\alpha \in \mathbb{Q}$ and $\beta \notin \mathbb{Q}$. Now assume for contradiction that $\alpha\beta \in \mathbb{Q}$. Since $\alpha \in \mathbb{Q}$ and $\alpha \neq 0$ we can write

$$\alpha = \frac{a}{b}$$

for some **nonzero** integers $a, b \in \mathbb{Z}$. Then since $\alpha\beta \in \mathbb{Q}$ we can write

$$\alpha\beta = \frac{c}{d}$$

for some integers $c, d \in \mathbb{Z}$. Since $\beta \notin \mathbb{Q}$ we know that $\beta \neq 0$, and hence $\alpha\beta \neq 0$, so that c and d are **nonzero**. Finally, we have

$$\beta = \frac{\alpha\beta}{\alpha} = \frac{c/d}{a/b} = \frac{bc}{ad}.$$

Since $ad \in \mathbb{Z}$ and $bc \in \mathbb{Z}$ with $ad \neq 0$ we conclude that $\beta \in \mathbb{Q}$, but this contradicts the fact that $\beta \notin \mathbb{Q}$. We conclude that the assumption $\alpha\beta \in \mathbb{Q}$ was false, hence $\alpha\beta \notin \mathbb{Q}$. $\square$

[Remark: It's okay if you didn't worry about the zeroness/nonzeroness of the integers. I didn't worry about it either. (In fact, I forgot to mention that $\alpha \neq 0$ in the statement of the problem.)]

(b) Use the result from part (a) to prove that $\sqrt{12} \notin \mathbb{Q}$. [Hint: You can assume without proof that $\sqrt{3} \notin \mathbb{Q}$. We proved this in class.]

Proof. First note that $\sqrt{12} = \sqrt{4 \cdot 3} = \sqrt{4}\sqrt{3} = 2\sqrt{3}$. Let $\alpha = 2$ and $\beta = \sqrt{3}$ so that $\alpha\beta = \sqrt{12}$. Since we know that $2 \in \mathbb{Q}$ and $\sqrt{3} \notin \mathbb{Q}$, the result of part (a) implies that $\sqrt{12} \notin \mathbb{Q}$. $\square$

Problem 4.

(a) Let P, Q, and R be logical statements. Use a truth table to prove that $P \Rightarrow (Q \vee R)$ is logically equivalent to $(P \wedge \neg Q) \Rightarrow R$.

P	Q	R	$Q \vee R$	$P \Rightarrow (Q \vee R)$	$\neg Q$	$P \wedge \neg Q$	$(P \wedge \neg Q) \Rightarrow R$
T	T	T	T	T	F	F	T
T	T	F	T	T	F	F	T
T	F	T	T	T	T	T	T
T	F	F	F	F	T	T	F
F	T	T	T	T	F	F	T
F	T	F	T	T	F	F	T
F	F	T	T	T	T	F	T
F	F	F	F	T	T	F	T

(b) Now consider two integers $m, n \in \mathbb{Z}$. Use the result from part (a) to prove that

$$\text{“}m \text{ is odd”} \Rightarrow \text{“}n \text{ is odd or } m^2 + n^2 \text{ is odd”}.$$

Proof. Let $P =$“m is odd”, $Q =$“n is odd”, and $R =$“$m^2 + n^2$ is odd”. We want to prove that $P \Rightarrow (Q \vee R)$. By part (a) it is enough to prove that $(P \wedge \neg Q) \Rightarrow R$, in other words,

$$\text{“}m \text{ is odd and } n \text{ is even”} \Rightarrow \text{“}m^2 + n^2 \text{ is odd”}.$$

So assume that m is odd and n is even; i.e., assume we can write $m = 2k$ and $n = 2\ell + 1$ for some integers $k, \ell \in \mathbb{Z}$. In this case we have

$$\begin{aligned} m^2 + n^2 &= (2k)^2 + (2\ell + 1)^2 \\ &= (4k^2) + (4\ell^2 + 4\ell + 1) \\ &= 2(2k^2 + 2\ell^2 + 2\ell) + 1, \end{aligned}$$

which is odd. $\square$