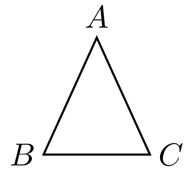
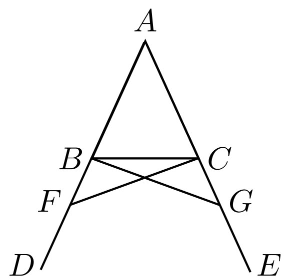
Math 230 E **Fall 2013**
Homework 1 Solutions **Drew Armstrong**

Problem 1. Proposition I.5 in Euclid has acquired the name *pons asinorum*, which translates as "bridge of asses" or "bridge of fools". Apparently, many students never got past this proposition. (I would say that the *pons asinorum* in today's curriculum is addition of fractions.) The proposition says the following: Consider a triangle $\triangle ABC$. **If** the side lengths $\overline{AB}$ and $\overline{AC}$ are equal (i.e. the triangle is *isosceles*), **then** the angles $\angle ABC$ and $\angle ACB$ are equal.



Your assignment is to **look up Euclid's proof and tell it to me.**

Proof. I will quote Euclid and try not to clean it up very much.

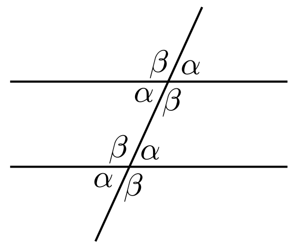


Extend the edge AB to some point D and the edge AC to some point E (Postulate 2). Choose an arbitrary point F on the segment BD (what allows us to do this—I'm not sure). Find the point G on AE such that the length of AG equals the length of AF (Proposition I.3). (Here we assumed that AE was long enough, but we could always make it longer if necessary.) Construct the segments FC and GB (Postulate 1). Since the side-angle-side triples $\overline{AF}, \angle FAC, \overline{AC}$ and $\overline{GA}, \angle GAB, \overline{AB}$ are equal, we conclude that the triangles $\triangle FAC$ and $\triangle GAB$ are congruent (Proposition I.4). In particular, we have $\angle ABG = \angle ACF$, $\angle AFC = \angle AGB$, and $\overline{FC} = \overline{GB}$.

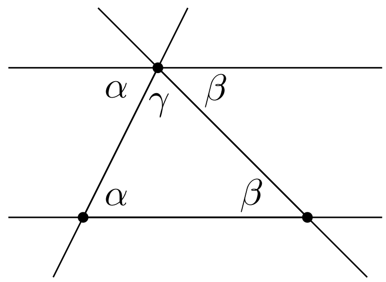
Next observe that $\overline{BF} = \overline{AF} - \overline{AB} = \overline{AG} - \overline{AC} = \overline{CG}$ (Common Notion 3). Since we also know that $\overline{FC} = \overline{GB}$ and $\angle BFC = \angle AFC = \angle AGB = \angle CGB$, we conclude that the triangles $\triangle FBC$ and $\triangle GCB$ are congruent (Proposition I.4). In particular we have $\angle BCF = \angle CBG$. Finally, we conclude that $\angle ABC = \angle ABG - \angle CBG = \angle ACF - \angle BCF = \angle ACB$ (Common Notion 3). $\square$

[Do you like this proof? Do you see why this is the place where many people got stuck?]

Problem 2. **Prove** that the interior angles of any triangle sum to 180°. You may use the following two facts without justification. **Fact 1:** Given a line ℓ and a point p not on ℓ, **it is possible** to draw a line through p parallel to ℓ. **Fact 2:** If a line falls on two parallel lines, then the corresponding angles are equal, as in the following figure.

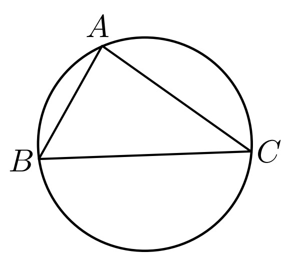


Proof. Consider a triangle and label its interior angles α, β and γ.

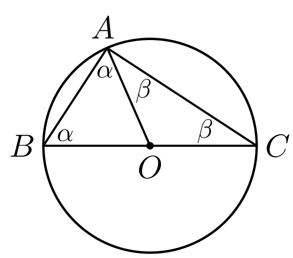


By **Fact 1** we may draw a line parallel to the side with angles α and β through the point at angle γ. Then by **Fact 2** we know that the angles next to γ on the same side of the line are equal to α and β. We conclude that α, β and γ add up to a straight line, i.e. $180°$. □

Problem 3. Prove Thales' Theorem, which says the following. Consider a triangle $\triangle ABC$ inscribed in a circle. **If** line segment BC is a diameter of the circle, **then** angle $\angle BAC$ is a right angle. You may quote the results from Problems 1 and 2.



Proof. Assume that BC is a diameter of the circle. Then (by definition, if you like) it passes through the center, which we label O. Consider the triangles $\triangle OAB$ and $\triangle OAC$.



Since segments OA, OB and OC all have the same length (they are all radii of the circle), the two triangles are isosceles. We conclude from Problem 1 the angles $\angle OAB$ and $\angle OBA$ are equall (call them α) and the angles $\angle OAC$ and $\angle OCA$ are equal (call them β). Thus the interior angles of $\triangle ABC$ are α, β and $\alpha+\beta$. By Problem 2 we know that these sum to 180°, hence

$$\begin{aligned} \alpha+\beta+(\alpha+\beta) &= 180^\circ \\ 2\alpha+2\beta &= 180^\circ \\ 2(\alpha+\beta) &= 180^\circ \\ \alpha+\beta &= 90^\circ, \end{aligned}$$

as desired. $\square$

Problem 4. The dot product of vectors $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is defined by $\mathbf{u}\cdot\mathbf{v} := u_1v_1 + u_2v_2 + \cdots + u_nv_n$. The length $\|\mathbf{u}\|$ of a vector $\mathbf{u}$ is defined by $\|\mathbf{u}\|^2 := \mathbf{u}\cdot\mathbf{u}$.

(a) Prove the formula $\|\mathbf{u}-\mathbf{v}\|^{\mathbf{2}} = \|\mathbf{u}\|^{\mathbf{2}} + \|\mathbf{v}\|^{\mathbf{2}} - \mathbf{2}(\mathbf{u}\cdot\mathbf{v})$.

(b) Use this formula together with the 2D Pythagorean Theorem **and its converse** to prove the following statement:

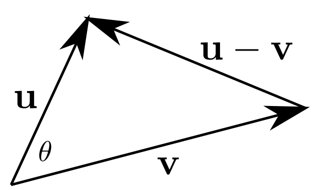
"the vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if and only if $\mathbf{u}\cdot\mathbf{v} = 0$."

[Hint: Where is the triangle? Recall that you must prove both directions of the *if and only if* statement separately.]

Proof. To prove part (a) we note that by definition of $\|\mathbf{u}-\mathbf{v}\|$ we have

$$\begin{aligned} \|\mathbf{u}-\mathbf{v}\|^2 = (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) &= (u_1-v_1)^2 + \cdots + (u_n-v_n)^2 \\ &= (u_1^2+v_1^2-2u_1v_1) + \cdots + (u_n^2+v_n^2-2u_nv_n) \\ &= (u_1^2+\cdots+u_n^2) + (v_1^2+\cdots+v_n^2) - 2(u_1v_1+\cdots+u_nv_n) \\ &= \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} - 2(\mathbf{u}\cdot\mathbf{v}) \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2(\mathbf{u}\cdot\mathbf{v}). \end{aligned}$$

To prove part (b) note that vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}-\mathbf{v}$ form a triangle as in the following figure. Let θ denote the angle between $\mathbf{u}$ and $\mathbf{v}$.

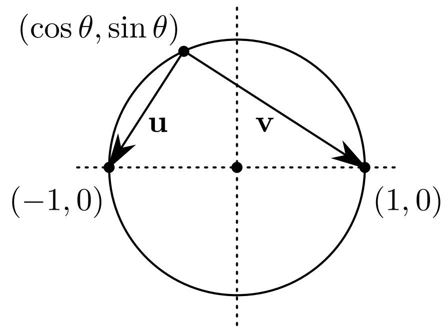


We wish to show that $\theta = 90^\circ$ **if and only if** $\mathbf{u}\cdot\mathbf{v} = 0$.

First we will show that $\theta = 90^\circ$ implies $\mathbf{u}\cdot\mathbf{v} = 0$. Assume that $\theta = 90^\circ$. Then the Pythagorean Theorem implies that $\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$, or $\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2 = 0$. But the result of part (a) says that $\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2 = 2(\mathbf{u}\cdot\mathbf{v})$. Combining the two equations gives $2(\mathbf{u}\cdot\mathbf{v}) = 0$, or $\mathbf{u}\cdot\mathbf{v} = 0$.

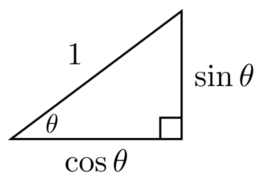
Next we will show that $\mathbf{u}\cdot\mathbf{v} = 0$ implies $\theta = 90^\circ$. Assume that $\mathbf{u}\cdot\mathbf{v} = 0$. Then the result of part (a) says that $\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2(0) = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$. Under these conditions, the **converse** of the Pythagorean Theorem implies that $\theta = 90^\circ$, as desired. $\square$

Problem 5. Use vectors to give an analytic proof of Thales' Theorem. [Hint: You may assume that your circle is the unit circle in the Cartesian plane. You may assume that $B = (-1, 0)$, $C = (1, 0)$, and $A = (\cos\theta, \sin\theta)$ for some angle θ. Consider the vectors $\mathbf{u} = A - B$ and $\mathbf{v} = A - C$. Compute the dot product $\mathbf{u} \cdot \mathbf{v}$.]



Proof. Note that the angle $\angle BAC$ is the same as the angle between the vector $\mathbf{u}$ and $\mathbf{v}$. By Problem 4(b) it is enough to show that $\mathbf{u} \cdot \mathbf{v} = 0$.

First, recall that for any angle θ we have $\cos^2\theta + \sin^2\theta = 1$. To see this, first assume that θ is less than $90°$ and consider the right triangle with hypotenuse of length 1 and angle θ.



The other sides of the triangle have length $\cos\theta$ and $\sin\theta$ (by definition of cos and sin) hence the Pythagorean Theorem says that $\cos^2\theta + \sin^2\theta = 1^2 = 1$. [Thinking Question: How can you prove this for angles θ larger than $90°$? Do you know the definition of $\cos\theta$ and $\sin\theta$ in that case?]

Finally, note that $\mathbf{u} = (-1 - \cos\theta, -\sin\theta)$ and $\mathbf{v} = (1 - \cos\theta, -\sin\theta)$. Then we have

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= (1 - \cos\theta, -\sin\theta) \cdot (-\cos\theta - 1, -\sin\theta) \\
&= (1 - \cos\theta)(-1 - \cos\theta) + (-\sin\theta)(-\sin\theta) \\
&= (-1 + \cos^2\theta) + \sin^2\theta \\
&= -1 + (\cos^2\theta + \sin^2\theta) \\
&= -1 + 1 \\
&= 0,
\end{aligned}$$

as desired. $\square$