Problem 1. Proposition I.5 in Euclid has acquired the name \textit{pons asinorum}, which translates as “bridge of asses” or “bridge of fools”. Apparently, many students never got past this proposition. (I would say that the \textit{pons asinorum} in today’s curriculum is addition of fractions.)

The proposition says the following: Consider a triangle $\triangle ABC$. If the side lengths $AB$ and $AC$ are equal (i.e. the triangle is \textit{isosceles}), then the angles $\angle ABC$ and $\angle ACB$ are equal.

Your assignment is to look up Euclid’s proof and tell it to me.

Proof. I will quote Euclid and try not to clean it up very much.

\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (1,1) node[above left] {$A$} -- (1,0) node[below right] {$C$} -- (0,0);
\end{tikzpicture}
\end{center}

Extend the edge $AB$ to some point $D$ and the edge $AC$ to some point $E$ (Postulate 2). Choose an arbitrary point $F$ on the segment $BD$ (what allows us to do this—I’m not sure). Find the point $G$ on $AE$ such that the length of $AG$ equals the length of $AF$ (Proposition I.3). (Here we assumed that $AE$ was long enough, but we could always make it longer if necessary.) Construct the segments $FC$ and $GB$ (Postulate 1). Since the side-angle-side triples $AF, \angle FAC, AC$ and $GA, \angle GAB, AB$ are equal, we conclude that the triangles $\triangle FAC$ and $\triangle GAB$ are congruent (Proposition I.4). In particular, we have $\angle ABG = \angle ACF$, $\angle AFC = \angle AGB$, and $FC = GB$.

Next observe that $BF = AF - AB = AG - AC = CG$ (Common Notion 3). Since we also know that $FC = GB$ and $\angle BFC = \angle AFC = \angle AGB = \angle CGB$, we conclude that the triangles $\triangle FBC$ and $\triangle GCB$ are congruent (Proposition I.4). In particular we have $\angle BCF = \angle CGB$. Finally, we conclude that $\angle ABC = \angle ABG - \angle CGB = \angle ACF - \angle BCF = \angle ACB$ (Common Notion 3).

[Do you like this proof? Do you see why this is the place where many people got stuck?]

Problem 2. Prove that the interior angles of any triangle sum to $180^\circ$. You may use the following two facts without justification. Fact 1: Given a line $\ell$ and a point $p$ not on $\ell$, it is possible to draw a line through $p$ parallel to $\ell$. Fact 2: If a line falls on two parallel lines, then the corresponding angles are equal, as in the following figure.
Proof. Consider a triangle and label its interior angles $\alpha$, $\beta$ and $\gamma$.

By Fact 1 we may draw a line parallel to the side with angles $\alpha$ and $\beta$ through the point at angle $\gamma$. Then by Fact 2 we know that the angles next to $\gamma$ on the same side of the line are equal to $\alpha$ and $\beta$. We conclude that $\alpha$, $\beta$ and $\gamma$ add up to a straight line, i.e. $180^\circ$. \hfill \Box

**Problem 3. Prove** Thales’ Theorem, which says the following. Consider a triangle $\triangle ABC$ inscribed in a circle. If line segment $BC$ is a diameter of the circle, then angle $\angle BAC$ is a right angle. You may quote the results from Problems 1 and 2.

Proof. Assume that $BC$ is a diameter of the circle. Then (by definition, if you like) it passes through the center, which we label $O$. Consider the triangles $\triangle OAB$ and $\triangle OAC$.

Since segments $OA$, $OB$ and $OC$ all have the same length (they are all radii of the circle), the two triangles are isosceles. We conclude from Problem 1 the angles $\angle OAB$ and $\angle OBA$ are equal (call them $\alpha$) and the angles $\angle OAC$ and $\angle OCA$ are equal (call them $\beta$). Thus the
interior angles of \( \triangle ABC \) are \( \alpha, \beta \) and \( \alpha + \beta \). By Problem 2 we know that these sum to 180°, hence

\[
\alpha + \beta + (\alpha + \beta) = 180^\circ \\
2\alpha + 2\beta = 180^\circ \\
2(\alpha + \beta) = 180^\circ \\
\alpha + \beta = 90^\circ,
\]

as desired. \( \square \)

**Problem 4.** The dot product of vectors \( u = (u_1, u_2, \ldots, u_n) \) and \( v = (v_1, v_2, \ldots, v_n) \) is defined by

\[
u \cdot v := u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.
\]

The length \( \|u\| \) of a vector \( u \) is defined by \( \|u\|^2 := u \cdot u \).

(a) Prove the formula

\[
\|u − v\|^2 = \|u\|^2 + \|v\|^2 - 2(u \cdot v).
\]

(b) Use this formula together with the 2D Pythagorean Theorem and its converse to prove the following statement:

"the vectors \( u \) and \( v \) are perpendicular if and only if \( u \cdot v = 0.\"

[Hint: Where is the triangle? Recall that you must prove both directions of the if and only if statement separately.]

**Proof.** To prove part (a) we note that by definition of \( \|u − v\| \) we have

\[
\|u − v\|^2 = (u − v) \cdot (u − v) = (u_1 − v_1)^2 + \cdots + (u_n − v_n)^2 \\
= (u_1^2 + v_1^2 − 2u_1 v_1) + \cdots + (u_n^2 + v_n^2 − 2u_n v_n) \\
= (u_1^2 + \cdots + u_n^2) + (v_1^2 + \cdots + v_n^2) − 2(u_1 v_1 + \cdots + u_n v_n) \\
= u \cdot u + v \cdot v − 2(u \cdot v) \\
= \|u\|^2 + \|v\|^2 − 2(u \cdot v).
\]

To prove part (b) note that vectors \( u, v, \) and \( u − v \) form a triangle as in the following figure. Let \( \theta \) denote the angle between \( u \) and \( v \).

![Diagram of triangle with vectors u, v, and u-v]

We wish to show that \( \theta = 90^\circ \) if and only if \( u \cdot v = 0. \)

**First** we will show that \( \theta = 90^\circ \) implies \( u \cdot v = 0. \) Assume that \( \theta = 90^\circ \). Then the Pythagorean Theorem implies that \( \|u − v\|^2 = \|u\|^2 + \|v\|^2 \), or \( \|u\|^2 + \|v\|^2 − \|u − v\|^2 = 0. \) But the result of part (a) says that \( \|u\|^2 + \|v\|^2 − \|u − v\|^2 = 2(u \cdot v). \) Combining the two equations gives \( 2(u \cdot v) = 0, \) or \( u \cdot v = 0. \)

**Next** we will show that \( u \cdot v = 0 \) implies \( \theta = 90^\circ. \) Assume that \( u \cdot v = 0. \) Then the result of part (a) says that \( \|u − v\|^2 = \|u\|^2 + \|v\|^2 − 2(0) = \|u\|^2 + \|v\|^2. \) Under these conditions, the converse of the Pythagorean Theorem implies that \( \theta = 90^\circ, \) as desired. \( \square \)
**Problem 5.** Use vectors to give an analytic proof of Thales' Theorem. [Hint: You may assume that your circle is the unit circle in the Cartesian plane. You may assume that $B = (-1, 0)$, $C = (1, 0)$, and $A = (\cos \theta, \sin \theta)$ for some angle $\theta$. Consider the vectors $u = A - B$ and $v = A - C$. Compute the dot product $u \cdot v$.]

\[
\begin{aligned}
& (\cos \theta, \sin \theta) \\
& \downarrow \quad \downarrow \\
& u \quad v \\
& (-1, 0) \quad (1, 0)
\end{aligned}
\]

*Proof. Note that the angle $\angle BAC$ is the same as the angle between the vector \( u \) and \( v \). By Problem 4(b) it is enough to show that $u \cdot v = 0$.

First, recall that for any angle $\theta$ we have $\cos^2 \theta + \sin^2 \theta = 1$. To see this, first assume that $\theta$ is less than $90^\circ$ and consider the right triangle with hypotenuse of length 1 and angle $\theta$.

The other sides of the triangle have length $\cos \theta$ and $\sin \theta$ (by definition of $\cos$ and $\sin$) hence the Pythagorean Theorem says that $\cos^2 \theta + \sin^2 \theta = 1^2 = 1$. [Thinking Question: How can you prove this for angles $\theta$ larger than $90^\circ$? Do you know the definition of $\cos \theta$ and $\sin \theta$ in that case?]

Finally, note that $u = (-1 - \cos \theta, -\sin \theta)$ and $v = (1 - \cos \theta, -\sin \theta)$. Then we have

\[
\begin{aligned}
u \cdot v &= (1 - \cos \theta, -\sin \theta) \cdot (-\cos \theta - 1, -\sin \theta) \\
&= (1 - \cos \theta)(-1 - \cos \theta) + (-\sin \theta)(-\sin \theta) \\
&= (-1 + \cos^2 \theta) + \sin^2 \theta \\
&= -1 + (\cos^2 \theta + \sin^2 \theta) \\
&= -1 + 1 \\
&= 0,
\end{aligned}
\]

as desired. □