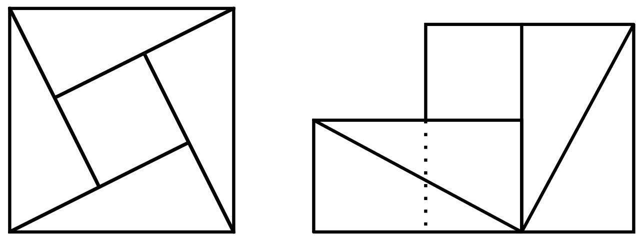
Math 230 D **Fall 2012**
Homework 1 **Drew Armstrong**

Problem 1. In the *Lilavati*, the Indian mathematician Bhaskara (1114–1185) gave a one-word proof of the Pythagorean theorem. He said: "Behold!"



Add words to the proof. Your goal is to persuade a high school student who claims he/she doesn't "get it". **Avoid algebra if possible.** (Sorry, the two pictures are not quite to scale.) [Hint: The dotted line is not in Bhaskara's figure. I added it as a suggestion.]

I will present two solutions. In both we label the short, medium, and long sides of the triangle by a, b, and c, respectively.

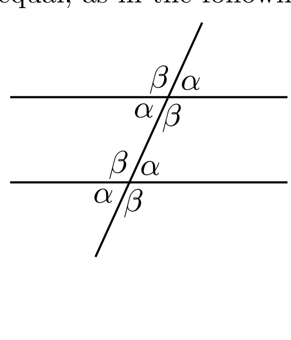
Solution 1. This is the solution that Bhaskara had in mind. There are three important observations: **(1)** The two figures have equal area because they are made out the same five pieces; i.e. four copies of the triangle and one square of side length $b - a$. **(2)** The figure on the left is a square of area c^2. **(3)** The figure on the right is divided by the dotted line into two squares, the smaller of area a^2 and the larger of area b^2. (To convince your high school pupil of this you should probably add labels to the right figure.) Combining the observations we conclude that $c^2 = a^2 + b^2$.

Solution 2. This solution is modern, and it **ignores** the right figure. This is definitely not what Bhaskara had in mind. Note that there are two ways to compute the area of the left figure. On one hand, it is a square of area c^2. On the other hand, it is formed from a square of area $(b-a)^2$ and four triangles of area $ab/2$. We conclude that

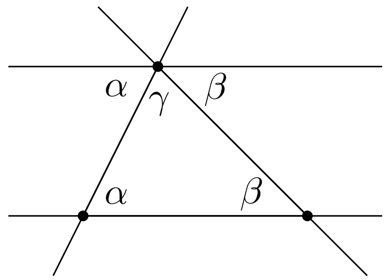
$$c^2 = (b-a)^2 + 4(ab/2) = a^2 + b^2 - 2ab + 2ab = a^2 + b^2.$$

[Note: Both solutions are acceptable but I much prefer the first. This is because we are trying to convince a high school student. Solution 1 is based on geometric intuition. Solution 2 is based on the algebra fact $(b-a)^2 = a^2 + b^2 - 2ab$, which the student might not "feel" to be true. To truly convince the student you would also have to "prove" this fact.]

Problem 2. Prove that the interior angles of any triangle sum to 180°. You may use the following two facts without justification. **Fact 1:** Given a line ℓ and a point p not on ℓ, **it is possible** to draw a line through p parallel to ℓ. **Fact 2:** If a line falls on two parallel lines, then the corresponding angles are equal, as in the following figure.



Proof. Draw a triangle with interior angles α, β, γ, as in the figure. By **Fact 1** we may draw a line parallel to the side $\alpha\beta$ through the point at γ. Then by **Fact 2** we know that the other angles beside γ are are α and β. Hence α, β, and γ add up to a straight line. $\square$



Problem 3. Prove that $\sqrt{3}$ is not a fraction, in two steps.

(a) First **prove** the following lemma: Given a whole number n, if n^2 is a multiple of 3, then so is n. [Hint: Use the contrapositive, and note that there are two ways for n to be "**not** divisible" by 3.]

(b) Use the method of contradiction to **prove** that $\sqrt{3}$ is not a fraction. Quote your lemma in the proof.

Lemma: If n^2 is a multiple of 3 then so is n.

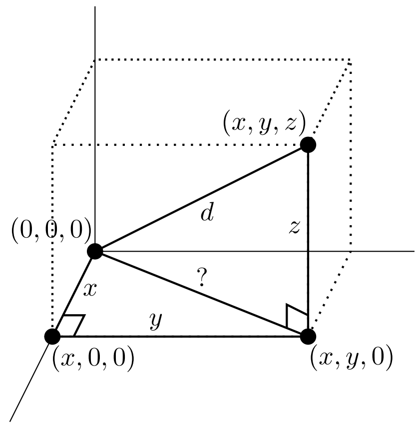
Proof: We will show the contrapositive statement — that if n is **not** a multiple of 3 then **neither** is n^2 — which is logically equivalent. So suppose that n is not a multiple of 3. There are two cases: **(1)** If $n = 3k+1$ for some k, then $n^2 = (3k+1)^2 = 9k^2+6k+1 = 3(3k^2+2k)+1$ is not a multiple of 3. (It leaves remainder 1 when divided by 3.) **(2)** If $n = 3k+2$ for some k, then $n^2 = (3k+2)^2 = 9k^2+12k+4 = 9k^2+12k+3+1 = 3(3k^2+4k+1)+1$ is also not a multiple of 3. $\square$

Theorem: $\sqrt{3}$ is not a ratio of whole numbers.

Proof: Suppose for contradiction that $\sqrt{3} = a/b$ for whole numbers a, b. After dividing out common factors we may assume that a and b have no common factor (other than ± 1). (We say they are "coprime".) Square both sides to get $3 = a^2/b^2$ and then multiply by b^2 to get $a^2 = 3b^2$. Since a^2 is a multiple of 3 the Lemma implies that $a = 3k$ for some k. But then $9k^2 = a^2 = 3b^2$ and dividing by 3 gives $b^2 = 3k^2$. The Lemma now implies that b is a multiple of 3. To summarize, we have shown that a and b are both divisible by 3, but this **contradicts** the fact that a, b are coprime. Hence our original assumption — that $\sqrt{3}$ is a ratio of whole numbers — must be false. $\square$

Problem 4. **Use** the 2D Pythagorean Theorem to **prove** the 3D Pythagorean Theorem. That is, prove that the distance between points $(0,0,0)$ and (x,y,z) equals $\sqrt{x^2+y^2+z^2}$. (Hint: There are two triangles involved.)

Let d denote the distance from $(0,0,0)$ to (x,y,z) and let ? denote the distance from $(0,0,0)$ to $(x,y,0)$ as in the following diagram.



Note that the triangle with side lengths $?, z, d$ is a right triangle, hence the 2D Pythagorean Theorem implies $d^2 = ?^2 + z^2$. Then note that the triangle with side lengths $x, y, ?$ is a right triangle, hence $?^2 = x^2 + y^2$. Combining the two equations gives $d^2 = x^2 + y^2 + z^2$, as desired. $\square$

Problem 5. The dot product of vectors $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is defined by $\mathbf{u} \cdot \mathbf{v} := u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$. The length $\|\mathbf{u}\|$ of a vector $\mathbf{u}$ is defined by $\|\mathbf{u}\|^2 := \mathbf{u} \cdot \mathbf{u}$.

(a) Prove the formula $\|\mathbf{u} - \mathbf{v}\|^{\mathbf{2}} = \|\mathbf{u}\|^{\mathbf{2}} + \|\mathbf{v}\|^{\mathbf{2}} - \mathbf{2}(\mathbf{u} \cdot \mathbf{v})$.

(b) Use this formula together with the 2D Pythagorean Theorem **and its converse** to prove the following statement:

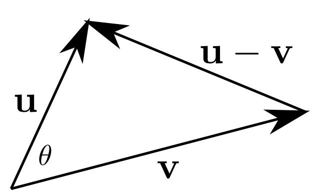
"the vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if and only if $\mathbf{u} \cdot \mathbf{v} = 0$."

(Hint: Where is the triangle?)

Proof: To prove part (a) we note that

$$
\begin{aligned}
\|\mathbf{u} - \mathbf{v}\|^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) &= (u_1 - v_1)^2 + \cdots + (u_n - v_n)^2 \\
&= (u_1^2 + v_1^2 - 2u_1 v_1) + \cdots + (u_n^2 + v_n^2 - 2u_n v_n) \\
&= (u_1^2 + \cdots + u_n^2) + (v_1^2 + \cdots + v_n^2) - 2(u_1 v_1 + \cdots + u_n v_n) \\
&= \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - 2(\mathbf{u} \cdot \mathbf{v}) \\
&= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2(\mathbf{u} \cdot \mathbf{v}).
\end{aligned}
$$

To prove part (b) note that vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} - \mathbf{v}$ form a triangle as in the following figure. Let θ denote the angle between $\mathbf{u}$ and $\mathbf{v}$.



We wish to show that $\theta = 90°$ **if and only if** $\mathbf{u} \cdot \mathbf{v} = 0$. **First** we will show that $\theta = 90°$ implies $\mathbf{u} \cdot \mathbf{v} = 0$. So suppose that $\theta = 90°$. Then the Pythagorean Theorem implies that $\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$. Combining this with part (a) yields $\mathbf{u} \cdot \mathbf{v} = 0$. **Next** we will show that $\mathbf{u} \cdot \mathbf{v} = 0$ implies $\theta = 90°$. So suppose that $\mathbf{u} \cdot \mathbf{v} = 0$. Then part (a) implies that $\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$. Finally, the **converse** Pythagorean Theorem implies that $\theta = 90°$, as desired. $\square$