Math 230 E — Fall 2011
Exam 3 — Fri Dec 2 — Drew Armstrong

There are 4 problems, worth 6 points each. There is 1 bonus point for writing your name. This is a closed book test. Anyone caught cheating will receive a score of **zero**.

1. [6 points]

(a) How many words are there with n letters from the alphabet $\{a, b\}$?

$$\boxed{2^n}$$

(b) How many words are there with k "a"s and $n-k$ "b"s?

$$\boxed{\binom{n}{k} = \frac{n!}{k!(n-k)!}}$$

(c) How many words are there with k "a"s, ℓ "b"s and $n-k-\ell$ "c"s?

$$\boxed{\binom{n}{k, \ell, n-k-\ell} = \frac{n!}{k!\ell!(n-k-\ell)!}}$$

(d) How many words are there with n letters from the alphabet $\{a, b, c\}$?

$$\boxed{3^n}$$

(e) **Accurately state** the Trinomial Theorem. This can we said in a few ways. For example:

$$(a+b+c)^n \sum_{\substack{i,j,k \in \mathbb{N} \\ i+j+k=n}} \frac{n!}{i!j!k!} a^i b^j c^k$$

2. [6 points] Recall that the Fibonacci numbers are defined by

$$f(n) := \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ f(n-1) + f(n-2) & \text{if } n \geq 2. \end{cases}$$

Let $P(n)$ be the statement " $f(n) < 2^n$ ".

(a) Verify a sufficient number of base cases.

Note that $P(0) =$ "$f(0) = 0 < 1 = 2^0$" $= T$ and $P(1) =$ "$f(1) = 1 < 2 = 2^1$" $= T$.

(b) Let $k \geq 2$ and state your induction hypothesis.

Fix $k \geq 2$ and assume that for all $0 \leq n \leq k$ we have $P(n) =$ "$f(n) < 2^n$" $= T$.

(c) Show that your induction hypothesis implies that $P(k+1)$ is true.

In this case, note that

$$f(k+1) = f(k) + f(k-1) \leq 2^k + 2^{k-1} < 2^k + 2^k = 2^{k+1},$$

hence $P(k+1)$ is true.

3. [6 points] Let $P(n)$ be the statement "For any sets $A_1, A_2, \ldots, A_n \subseteq U$ we have

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c."$$

The base case $P(2)$ is called De Morgan's Law and it is true. Prove by induction that $P(n)$ is true for all $n \geq 2$, in three steps.

(a) **Accurately state** the principle of induction.

Let $P : \mathbb{N} \to \{T, F\}$. If (1) $P(b) = T$ for some b and if (2) for all $k \geq b$ we have $P(k) \Rightarrow P(k+1)$, then $P(n) = T$ for all $n \geq b$.

(b) Let $k \geq 2$ and state your induction hypothesis.

Fix $k \geq 2$ and assume that $P(k) = T$; i.e. for all $A_1, \ldots, A_k \subseteq U$ we have

$$(A_1 \cup \cdots \cup A_k)^c = A_1^c \cap \cdots \cap A_k^c.$$

(c) Prove that your induction hypothesis implies that $P(k+1)$ is true.

Now consider any $A_1, \ldots, A_{k+1} \subseteq U$. In this case we have

$$\begin{aligned}(A_1 \cup \cdots \cup A_{k+1}) &= ((A_1 \cup \cdots \cup A_k) \cup A_{k+1})^c \\ &= (A_1 \cup \cdots \cup A_k)^c \cap A_{k+1}^c \\ &= A_1^c \cap \cdots \cap A_k^c \cap A_{k+1}^c.\end{aligned}$$

Hence $P(k+1) = T$.

4. [6 points] The following problems can be solved by manipulating the polynomial $(1+x)^n$ and plugging in appropriate values for x.

(a) For all $n \geq 1$ prove that $\displaystyle\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$.

By putting $x = 2$ in $(1+x)^n = \sum_{k=0}^{n} x^k \binom{n}{k}$ we get $3^n = \sum_{k=0}^{n} 2^k \binom{n}{k}$.

(b) For all $n \geq 1$ prove that $\displaystyle\sum_{k=0}^{n} k\, 3^{k-1} \binom{n}{k} = n\, 4^{n-1}$.

First differentiate both sides of $(1+x)^n = \sum_{k=0}^{n} x^k \binom{n}{k}$ by x to get $n(1+x)^{n-1} = \sum_{k=0}^{n} kx^{k-1} \binom{n}{k}$. Then put $x = 3$ to get $n4^{n-1} = \sum_{k=0}^{n} k3^{k-1} \binom{n}{k}$.

(c) Evaluate the sum $\displaystyle\sum_{k=0}^{n} k\, 3^k \binom{n}{k}$.

By part (b) we have

$$\sum_{k=0}^{n} k3^k \binom{n}{k} = 3 \left(\sum_{k=0}^{n} k3^{k-1} \binom{n}{k} \right) = 3(n4^{n-1}).$$