Math 230 D **Fall 2012**
Exam 1 — Fri Sept 21 **Drew Armstrong**

There are 4 problems, worth 5 points each. This is a closed book test. Anyone caught cheating will receive a score of **zero**.

Problem 1. Let P and Q be Boolean variables.

(a) Draw the truth table for the Boolean function $P \Rightarrow Q$.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

(b) Use a truth table to prove that $P \Rightarrow Q$ is logically equivalent to the Boolean function (NOT P) OR Q.

P	Q	$\text{NOT}\,P$	$(\text{NOT}\,P)\,\text{OR}\,Q$	$P \Rightarrow Q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

(c) Apply de Morgan's law and part (b) to find an expression for NOT $(P \Rightarrow Q)$.

$$\begin{aligned}
\text{NOT}\,(P \Rightarrow Q) &= \text{NOT}\,((\text{NOT}\,P)\,\text{OR}\,Q) \\
&= (\text{NOT}\,(\text{NOT}\,P))\,\text{AND}\,(\text{NOT}\,Q) \\
&= P\,\text{AND}\,(\text{NOT}\,Q).
\end{aligned}$$

Problem 2. Let P, Q and R be Boolean variables.

(a) Use a truth table to prove that the Boolean function $P \Rightarrow (Q \Rightarrow R)$ is logically equivalent to $X := (P\,\text{AND}\,(\text{NOT}\,R)) \Rightarrow (\text{NOT}\,Q)$.

P	Q	R	$Q \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$	$\text{NOT}\,R$	$P\,\text{AND}\,(\text{NOT}\,R)$	$\text{NOT}\,Q$	X
T	T	T	T	T	F	F	F	T
T	T	F	F	F	T	T	F	F
T	F	T	T	T	F	F	T	T
T	F	F	T	T	T	T	T	T
F	T	T	T	T	F	F	F	T
F	T	F	F	T	T	F	F	T
F	F	T	T	T	F	F	T	T
F	F	F	T	T	T	F	T	T

(b) Now let m and n be integers and consider the following statement: "If m is **odd**, then whenever mn is **even**, it follows that n is **even**." Define P, Q and R so this statement has the form $P \Rightarrow (Q \Rightarrow R)$.

Let $P =$"m is odd", $Q =$"mn is even" and $R =$"n is even."

(c) Use part (a) to prove the statment from part (b).

Proof. Instead of the statement $P \Rightarrow (Q \Rightarrow R)$, we will prove the statement $(P \text{ AND } (\text{NOT } R)) \Rightarrow (\text{NOT } Q)$, which is logically equivalent by part (a). That is, we will prove that for all integers $m, n \in \mathbb{Z}$ we have (m is odd and n is odd)$\Rightarrow$(mn is odd).

So suppose that m and n are both odd, say $m = 2k + 1$ and $n = 2\ell + 1$ for some integers $k, \ell \in \mathbb{Z}$. Then we have

$$\begin{aligned} mn &= (2k+1)(2\ell+1) \\ &= 4k\ell + 2k + 2\ell + 1 \\ &= 2(2k\ell + k + \ell) + 1, \end{aligned}$$

which is odd, as desired. $\square$

Problem 3. Let n be an integer and **assume** for the moment the following fact: "If n^2 is a multiple of 5, then so is n." Use this fact (i.e. **quote** it at the appropriate time) to **prove by contradiction** that $\sqrt{5}$ is **not** a ratio of integers.

Proof. Suppose for contradiction that $\sqrt{2}$ **is** a ratio of integers. Then we can write $\sqrt{2} = a/b$ where $a, b \in \mathbb{Z}$ are integers with no common divisor (i.e. the fraction is in "lowest terms"). Squaring both sides gives $2 = a^2/b^2$ and then multiplying by b^2 gives $a^2 = 2b^2$. Now we see that a^2 is even, and the FACT implies that a is even, say $a = 2k$. Substituting this into $a^2 = 2b^2$ gives $4k^2 = 2b^2$, or $2k^2 = b^2$. Hence b^2 is even, and the FACT implies that b itself is even. We now have that a and b are **both** even, but this contradicts the assumption that they have no common divisor. $\square$

Problem 4. Let n be an integer.

(a) What is the contrapositive of this statement?

"If n^2 is a multiple of 5, then so is n."

Let $P =$"n^2 is a multiple of 5" and let $Q =$"n is a multiple of 5." Then the contrapositive of $P \Rightarrow Q$ is $(\text{NOT } Q) \Rightarrow (\text{NOT } P)$, which says: "If n is **not** a multiple of 5, then **neither** is n^2."

(b) If you want to prove the statement from (a), the argument will break into **four cases**. Tell me what the four cases are.

To prove the statement, we assume that n is **not** a multiple of 5. There are four ways this can happen: (1) $n = 5k + 1$ for some $k \in \mathbb{Z}$, (2) $n = 5k + 2$ for some $k \in \mathbb{Z}$, (3) $n = 5k + 3$ for some $k \in \mathbb{Z}$, and (4) $n = 5k + 4$ for some $k \in \mathbb{Z}$.

(c) [**1 Bonus Point**] Finish the proof of the statement from (a). (Use the back of the page if necessary.)

Squaring n in each case gives: (1) $n^2 = 5(5k^2+2k)+1$, (2) $n^2 = 5(5k^2+4k)+4$, (3) $n^2 = 5(5k^2 + 6k + 1) + 4$, and (4) $n^2 = 5(5k^2 + 8k + 3) + 1$. In any case, we conclude that n^2 is **not** a multiple of 5, as desired.